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Research Article

An Effective Solution of the Cube-Root Truly Nonlinear Oscillator: Extended Iteration Procedure

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The cube-root truly nonlinear oscillator and the inverse cube-root truly nonlinear oscillator are the most meaningful and classical nonlinear ordinary differential equations on behalf of its various applications in science and engineering. Especially, the oscillators are used widely in the study of elastic force, structural dynamics, and elliptic curve cryptography. In this paper, we have applied modified Mickens extended iteration method to solve the cube-root truly nonlinear oscillator, the inverse cube-root truly nonlinear oscillator, and the equation of pendulum. Comparison is made among iteration method, harmonic balance method, He's amplitude-frequency formulation, He's homotopy perturbation method, improved harmonic balance method, and homotopy perturbation method. After comparison, we analyze that modified Mickens extended iteration method is more accurate, effective, easy, and straightforward. Also, the comparison of the obtained analytical solutions with the numerical results represented an extraordinary accuracy. The percentage error for the fourth approximate frequency of cube-root truly nonlinear oscillator is 0.006 and the percentage error for the fourth approximate frequency of inverse cube-root truly nonlinear oscillator is 0.12.

1. Introduction

Nonlinear systems are widespread around us. Nonlinear systems are widely involved in science, engineering, medical science, etc. So, research on the nonlinear systems has enriched science, engineering, medical science, etc. Research on nonlinear systems is complex and sensitive because most of the nonlinear systems suddenly change their characteristics due to some small changes of some valid parameters. The development of the theorems of dynamical systems has been derived by the modeling and formulating of nonlinear oscillators. Thus, nonlinear oscillators are one of the most important parts of nonlinear dynamical systems. Recently, many scientists have made significant improvement in finding a new mathematical tool which would be related to nonlinear dynamical systems whose understanding will rely not on analytic techniques but also on numerical and asymptotic methods. They set up many fruitful and potential methods to operate the nonlinear systems. There are many analytical methods to solve nonlinear oscillators such as perturbation [1–3]; standard and modified Poincare–Linstedt [4]; harmonic balance [5–7]; multiple scale [8]; homotopy perturbation [9–14]; modified He's homotopy perturbation [15]; He's frequency-amplitude formulation [16, 17]; cubication method [18, 19]; energy balance method [20]; He's energy balance method [21]; Mickens iteration method [7, 22–25]; Hu's iteration method [26]; Haque's iteration method [27–29]; Haque's extended iteration method [30–32]; variation iteration method [33]; homotopy analysis method [34]; finite element method and Akbari Ganji method [35]; and Taguchi method [36]. Besides, a few numbers of researchers have done work on the cube-root nonlinear oscillator using different methods such as the methods by Beléndez [9], Beléndez et al. [15], Ganji et al. [21], Mickens [25], Zhang [17], and Lim and Wu [6]. Among them, the proposed method is more easy, simple, and also valid for higher order approximation solutions. Perturbation method is the most widely utilized established method in which the nonlinear terms are considered as small. Mickens has developed a technique named harmonic balance (HB) method and further work has been done by Lim, Hu, Wu, and Alam, and so forth, for handling the oscillators in which nonlinear terms are strong. Mickens has developed another update technique named iteration method. The method of iteration is applicable for small as well as strong nonlinearities. Mickens has also developed the iteration method named extended iteration method; sometimes, this is faster than direct or simple iteration method. Further, the method of iteration and extended iteration has been developed by Lim, Hu, and Haque. Ikramul Haque et al. applied the proposed method to obtain the suitable periodic solution of nonlinear oscillators [30–32].

The purpose of this article is to exhibit an extended iteration method which gives us a simple path line for obtaining the approximate angular frequencies and corresponding analytic solutions and a significantly improved results of the "cube-root truly nonlinear oscillator (TNL)," " the inverse cube-root TNL oscillator," and "the pendulum equation."

2. The Method

The fundamental base of the method of extended iteration is to reconvey the prestage's solution to obtain the present stage's solution and its sequence of extensions. The most important concern is

(i) How to rearrange the preliminary shape of the given nonlinear oscillator so that it will help us to handle step by step iterations with simplification of a term including large number of harmonics.

(ii) How to reuse the obtained solutions for the case of harmonic terms so that there is no algebraic complexity.

The outline of the presented procedure is as follows:

(i)Step 1: considering the general form of oscillator by the following way:

$$\ddot{x} + f(x) = 0. \tag{1}$$

(ii) Step 2: setting initial conditions as

$$x(0) = A, \quad \dot{x}(0) = 0. \tag{2}$$

(iii) Step 3: making standard form by adding $\Omega^2 x$ on both sides, we have

$$\ddot{x} + \Omega^2 x = \Omega^2 x - f(x) \equiv G(x). \tag{3}$$

(iv) Step 4: formulizing the suitable iteration scheme as

$$\ddot{x}_{k+1} + \Omega_k^2 x_{k+1} = G(x_k) + G_x(x_0, \Omega_k)(x_k - x_0); \quad k = 1, 2, 3, \ldots. \tag{4}$$

(v) Step 5: setting initial supposition as

$$x_0(t) = A\cos(\Omega_0 t). \tag{5}$$

(vi) Step 6: setting iterated initial conditions as

$$x_{k+1}(0) = A, \quad \dot{x}_{k+1}(0) = 0. \tag{6}$$

(vii) Step 7: executing the successive iteration from $k = 1, 2, 3, \ldots$.

3. Solution Procedure of Cube-Root TNL Oscillator

Consider the cube-root TNL oscillator [25]

$$\ddot{x} + x^{1/3} = 0. \tag{7}$$

Now, introducing the term $\Omega^2 x$ on equation (7), then we have the standard form:

$$\ddot{x} + \Omega^2 x = \Omega^2 x - x^{1/3} = G(x, \Omega), \tag{8}$$

where $G(x, \Omega) = \Omega^2 x - x^{1/3}$.

Therefore, $G_x = \Omega^2 - (1/3)x^{-(2/3)}$.

Using equation (4), the iteration scheme of equation (8) is

$$\ddot{x}_{k+1} + \Omega_k^2 x_{k+1} = \left(\Omega_k^2 x_0 - x_0^{1/3}\right) + \left(\Omega_k^2 - \frac{1}{3}x_0^{-(2/3)}\right)(x_k - x_0). \tag{9}$$

First Iteration Step. Putting $k = 0$ in equation (9) and using initial supposition (5), we get

$$\ddot{x}_1 + \Omega_0^2 x_1 = \Omega_0^2 A\cos\theta - (A\cos\theta)^{1/3}. \tag{10}$$

Now, expanding the right hand side of equation (10), then it reduces to

$$\ddot{x}_1 + \Omega_0^2 x_1 = \left(\Omega_0^2 A - 1.1596A^{1/3}\right)\cos\theta + 0.231919A^{1/3}\cos 3\theta - 0.11596A^{1/3}\cos 5\theta. \tag{11}$$

To avoid secular term from equation (11), we obtain

$$\Omega_0 = \frac{1.076847}{A^{1/3}} = \Omega_{\mathrm{CR0}}, \text{ (say)}. \tag{12}$$

Then, equation (11) is transformed into

$$\ddot{x}_1 + \Omega_0^2 x_1 = 0.231919 A^{1/3} \cos 3\theta - 0.11596 A^{1/3} \cos 5\theta. \tag{13}$$

The required solution of equation (13) is

$$x_1(t) = A\left(1.0208398 - 0.0249998 \cos 3\theta + 0.00416 \cos 5\theta\right) = x_{\mathrm{CR1}}(t), \text{ (say)}. \tag{14}$$

This is known as the first iterated approximate solution of the oscillator (7).

Second Iteration Step. For the second level of iteration, we have

$$\ddot{x}_2 + \Omega_1^2 x_2 = \Omega_1^2 x_1 - \frac{1}{3} x_0^{-(2/3)} x_1 - \frac{2}{3} x_0^{1/3}. \tag{15}$$

Now, substituting the value of x_0 and x_1 from equations (5) and (14) into (15) and expanding the right side of equation (15), it reduces to

$$\begin{aligned} x_2 + \Omega_1^2 x_2 = {} & \left(1.0208398 A \Omega_1^2 - 1.169748 A^{1/3}\right) \cos\theta \\ & + \left(-0.0249998 A \Omega_1^2 + 0.246675 A^{1/3}\right) \cos 3\theta \\ & + \left(0.00416 A \Omega_1^2 - 0.122544 A^{1/3}\right) \cos 5\theta. \end{aligned} \tag{16}$$

To eliminate secular term, we get

$$\Omega_1 = \frac{1.070452}{A^{1/3}} = \Omega_{\mathrm{CR1}}, \text{ (say)}. \tag{17}$$

Then, equation (16) is transferred into

$$\ddot{x}_2 + \Omega_1^2 x_2 = 0.2180285 A^{1/3} \cos 3\theta - 0.117777 A^{1/3} \cos 5\theta. \tag{18}$$

The required solution of equation (18) is

$$\begin{aligned} x_2(t) = {} & A\left(1.019529 \cos\theta - 0.023812 \cos 3\theta \right. \\ & \left. + 0.004283 \cos 5\theta\right) = x_{\mathrm{CR2}}(t), \text{ (say)}. \end{aligned} \tag{19}$$

This is known as the second iterated approximate solution of the oscillator (7).

Third Iteration Step. For the third level of iteration, we have

$$\ddot{x}_3 + \Omega_2^2 x_2 = \Omega_2^2 x_2 - \frac{1}{3} x_0^{-2/3} x_2 - \frac{2}{3} x_0^{1/3}. \tag{20}$$

Now, substituting the value of x_0 and x_2 from equations (5) and (19) into (20) and expanding the right side of equation (20), it reduces to

$$\begin{aligned} \ddot{x}_3 + \Omega_2^2 x_3 = {} & \left(1.019529 A \Omega_2^2 - 1.169213 A^{1/3}\right) \cos\theta \\ & - \left(0.023812 A \Omega_2^2 - 0.245996 A^{1/3}\right) \cos 3\theta \\ & + \left(0.004283 A \Omega_2^2 - 0.122395 A^{1/3}\right) \cos 5\theta. \end{aligned} \tag{21}$$

To eliminate secular term, we get

$$\Omega_2 = \frac{1.070895}{A^{1/3}} = \Omega_{\mathrm{CR2}}, \text{ (say)}. \tag{22}$$

Then, equation (21) is transferred into

$$\ddot{x}_3 + \Omega_2^2 x_3 = 0.220496 A^{1/3} \cos 3\theta - 0.117808 A^{1/3} \cos 5\theta. \tag{23}$$

The required solution of equation (23) is

$$x_3(t) = A\left(1.021153 \cos\theta - 0.025737 \cos 3\theta + 0.004584 \cos 5\theta\right) = x_{\mathrm{CR3}}(t), \text{ (say)}. \tag{24}$$

This is known as the third iterated approximate solution of the oscillator (7).

Fourth Iteration Step. For the fourth level of iteration, we have

$$\ddot{x}_4 + \Omega_3^2 x_4 = \Omega_3^2 x_3 - \frac{1}{3} x_0^{(-2/3)} x_3 - \frac{2}{3} x_0^{1/3}. \tag{25}$$

Now, substituting the value of x_0 and x_3 from equations (5) and (24) into (25) and expanding the right side of equation (25), it reduces to

$$\begin{aligned} \ddot{x}_4 + \Omega_3^2 x_4 = {} & \left(1.021153 A \Omega_3^2 - 1.169943 A^{1/3}\right) \cos\theta \\ & - \left(0.025737 A \Omega_3^2 - 0.247131 A^{1/3}\right) \cos 3\theta \\ & + \left(0.004584 A \Omega_3^2 - 0.122892 A^{1/3}\right) \cos 5\theta. \end{aligned} \tag{26}$$

To eliminate secular term, we get

$$\Omega_3 = \frac{1.070378}{A^{1/3}} = \Omega_{\mathrm{CR3}}, \text{ (say)}. \tag{27}$$

Equations (12), (17), (22), and (27) represent the first, second, third, and fourth approximate frequencies of the cube-root TNL oscillator (7), respectively.

4. Solution of Inverse Cube-Root TNL Oscillator

Consider the inverse cube-root TNL oscillator [25]

$$\ddot{x} + x^{-(1/3)} = 0. \tag{28}$$

Applying the same iteration procedure, we obtain

$$\left.\begin{aligned} \Omega_{ICR0} &= \frac{1.194298}{A^{2/3}} \\ \Omega_{ICR1} &= \frac{1.146861}{A^{2/3}} \\ \Omega_{ICR2} &= \frac{1.158141}{A^{2/3}} \\ \Omega_{ICR3} &= \frac{1.156090}{A^{2/3}} \end{aligned}\right\}, \tag{29}$$

$$\left.\begin{aligned} x_{ICR1}(t) &= A(1.052312\cos\theta - 0.0249998\cos 3\theta + 0.00416\cos 5\theta) \\ x_{ICR2}(t) &= A(1.038587\cos\theta - 0.052771\cos 3\theta + 0.014184\cos 5\theta) \\ x_{ICR3}(t) &= A(1.041074\cos\theta - 0.054852\cos 3\theta + 0.013778\cos 5\theta) \end{aligned}\right\}. \tag{30}$$

Equation (29) represents the first, second, third, and fourth approximate frequencies of the inverse cube-root TNL oscillator (28) which are denoted by Ω_{ICR0}, Ω_{ICR1}, Ω_{ICR2}, and Ω_{ICR3}, respectively. Also, equation (30) represents the first, second, and third approximate analytic solutions of the inverse cube-root TNL oscillator (28) which are denoted by x_{ICR0}, x_{ICR1} and x_{ICR2}, respectively.

5. Equation of Pendulum

Consider the following equation as a pendulum equation where friction is neglected, as shown in Figure 1 [13].where $K^2 = (g/l)$, g is the gravitational acceleration, l is the length of the pendulum, and θ is the angle from the vertical to the pendulum.

$$\begin{aligned} \ddot{\theta} + K^2 \sin\theta &= 0, \\ \theta(0) &= A, \\ \dot{\theta}(0) &= 0, \end{aligned} \tag{31}$$

To solve the pendulum equation, we consider $\sin\theta \approx \theta - (1/6)\theta^3 + (1/120)\theta^5$, then equation (30) reduces to

$$\ddot{\theta} + K^2\left(\theta - \frac{1}{6}\theta^3 + \frac{1}{120}\theta^5\right) = 0, \quad \text{with } \theta(0) = A, \theta(0) = 0. \tag{32}$$

The iteration form of equation (32) is

$$\ddot{\theta} + \Omega^2 x = \Omega^2\theta - K^2\left(\theta - \frac{1}{6}\theta^3 + \frac{1}{120}\theta^5\right) = H(\theta, \Omega), \tag{33}$$

where

$$\begin{aligned} H(\theta,\Omega) &= \Omega^2\theta - K^2\left(\theta - \frac{1}{6}\theta^3 + \frac{1}{120}\theta^5\right), \\ H_\theta &= \frac{\partial H}{\partial\theta} = \Omega^2 - K^2\left(1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4\right). \end{aligned} \tag{34}$$

Applying the approximate method (6), we get

$$\begin{aligned} \ddot{\theta}_{k+1} + \Omega_k^2\theta_{k+1} = &\left(\Omega_k^2\theta_0 - K^2\left(\theta_0 - \frac{1}{6}\theta_0^3 + \frac{1}{120}\theta_0^5\right)\right. \\ &\left. + \left\{\Omega_k^2 - K^2\left(1 + \frac{1}{2}\theta_0^2 - \frac{1}{24}\theta_0^4\right)\right\}(\theta_k - \theta_0)\right). \end{aligned} \tag{35}$$

For the first iteration, we get

$$\ddot{\theta}_1 + \Omega_0^2\theta_1 = \Omega_0^2 A\cos\theta - K^2\left\{A\cos\theta - \frac{1}{6}(A\cos\theta)^3 + \frac{1}{120}(A\cos\theta)^5\right\}. \tag{36}$$

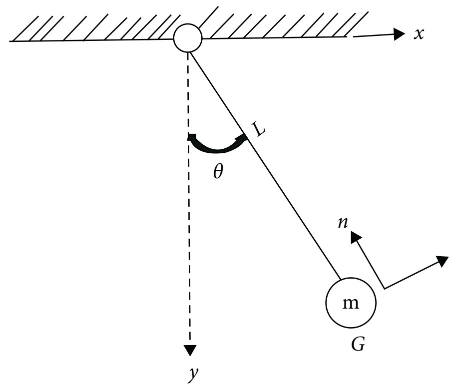

Figure 1: The simple pendulum.

Now, expanding the right side of equation (36) using a Fourier series, we get

$$\ddot{\theta}_1 + \Omega_0^2\theta_1 = \left\{\Omega_0^2 A - K^2\left(A - \frac{1}{8}A^3 + \frac{1}{192}A^5\right)\right\}\cos\theta$$
$$-K^2\left(-\frac{1}{24}A^3 + \frac{1}{384}A^5\right)\cos 3\theta - K^2\left(\frac{1}{1920}A^5\right)\cos 5\theta. \quad (37)$$

To avoid secular term, we obtain

$$\Omega_0 = K\left(1 - \frac{1}{8}A^2 + \frac{1}{192}A^4\right)^{1/2}. \quad (38)$$

This is the first approximate frequency of the oscillator. After simplification in equation (37), we have

$$\ddot{\theta}_1 + \Omega_0^2\theta_1 = -K^2\left(-\frac{1}{24}A^3 + \frac{1}{384}A^5\right)\cos 3\theta - K^2\left(\frac{1}{1920}A^5\right)\cos 5\theta. \quad (39)$$

Thus, the first iterated complete solution of the oscillator (31) is

$$\theta_1(t) = f_1\cos\theta + f_2\cos 3\theta + f_3\cos 5\theta, \quad (40)$$

where

$$f_1 = \frac{A - (23/192)A^3 + (7/1440)A^5}{\left(1 - (1/8)A^2 + (1/192)A^4\right)},$$
$$f_2 = \frac{-(1/24)A^3 + (1/384)A^5}{8\left(1 - (1/8)A^2 + (1/192)A^4\right)}, \quad (41)$$
$$f_3 = \frac{(1/1920)A^5}{24\left(1 - (1/8)A^2 + (1/192)A^4\right)}.$$

For the second iteration, we get

$$\ddot{\theta}_2 + \Omega_1^2\theta_2 = \Omega_1^2\theta_1 - K^2\left(-\frac{1}{2}\theta_0^2\theta_1 + \frac{1}{24}\theta_0^4\theta_1 + \frac{1}{3}\theta_0^3 - \frac{1}{30}\theta_0^5 + \theta_1\right), \quad (42)$$

$$\begin{aligned}\ddot{\theta}_2 + \Omega_1^2\theta_2 = {} & \Omega_1^2\left(f_1\cos\theta + f_2\cos\theta + f_3\cos\theta\right)\\ & - K^2\left(f_1\cos\theta + f_2\cos 3\theta + f_3\cos 5\theta\right)\\ & + \frac{1}{2}K^2A^2(\cos\theta)^2\left(f_1\cos\theta + f_2\cos 3\theta + f_3\cos 5\theta\right) - \frac{1}{3}K^2A^3(\cos\theta)^3\\ & - \frac{1}{24}K^2A^4(\cos\theta)^4\left(f_1\cos\theta + f_2\cos 3\theta + f_3\cos 5\theta\right) + \frac{1}{30}K^2A^5(\cos\theta)^5.\end{aligned} \quad (43)$$

Now, expanding the right side of equation (43) using truncation principle, we get

$$\ddot{\theta}_2 + \Omega_1^2 x_2 = \left\{\Omega_1^2 f_1 - K^2\left\{f_1 - \frac{3}{4}\left(-\frac{1}{3}A^3 + \frac{1}{2}A^2 f_1\right) - \frac{1}{8}A^2 f_2 - \frac{5}{8}\left(\frac{1}{30}A^5 - \frac{1}{24}A^4 f_1\right) + \frac{5}{384}A^4 f_2 + \frac{1}{384}A^4 f_3\right\}\right\}\cos\theta$$
$$+\left\{\Omega_1^2 f_2 - K^2\left\{f_2 + \frac{1}{4}\left(-\frac{1}{2}A^2 f_1 + \frac{1}{3}A^3\right) - \frac{1}{4}A^2 f_2 - \frac{1}{8}A^2 f_3 + \frac{5}{16}\left(\frac{1}{24}A^4 f_1 - \frac{1}{30}A^5\right) + \frac{1}{64}A^4 f_2 + \frac{1}{96}A^4 f_3\right\}\right\}\cos 3\theta$$
$$+\left\{\Omega_1^2 f_3 - K^2\left\{f_3 - \frac{1}{8}A^2 f_2 - \frac{1}{4}A^2 f_3 + \frac{1}{16}\left(\frac{1}{24}A^4 f_1 - \frac{1}{30}A^5\right) + \frac{1}{96}A^4 f_2 + \frac{1}{64}A^4 f_3\right\}\right\}\cos 5\theta. \quad (44)$$

To avoid secular term, we obtain

$$\Omega_1 = K\left[\begin{array}{c}\left(1 - \frac{3}{8}A^2 + \frac{5}{192}A^4\right) + \left(-\frac{1}{8}A^2 + \frac{5}{384}A^4\right)\frac{f_2}{f_1} \\ + \left(\frac{1}{384}A^4\right)\frac{f_3}{f_1} - \frac{1}{f_1}\left(-\frac{1}{4}A^3 + \frac{1}{48}A^5\right)\end{array}\right]^{1/2}. \quad (45)$$

This is the second approximate frequency of the oscillator.

After simplification in equation (44), we have

$$\ddot{x}_2 + \Omega_1^2 x_2 = \lambda_1 \cos 3\theta + \lambda_2 \cos 5\theta, \quad (46)$$

where

$$\lambda_1 = \left\{\Omega_1^2 f_2 - K^2\left\{f_2 + \frac{1}{4}\left(-\frac{1}{2}A^2 f_1 + \frac{1}{3}A^3\right) - \frac{1}{4}A^2 f_2 - \frac{1}{8}A^2 f_3 + \frac{5}{16}\left(\frac{1}{24}A^4 f_1 - \frac{1}{30}A^5\right) + \frac{1}{64}A^4 f_2 + \frac{1}{96}A^4 f_3\right\}\right\},$$
$$\lambda_2 = \left\{\Omega_1^2 f_3 - K^2\left\{f_3 - \frac{1}{8}A^2 f_2 - \frac{1}{4}A^2 f_3 + \frac{1}{16}\left(\frac{1}{24}A^4 f_1 - \frac{1}{30}A^5\right) + \frac{1}{96}A^4 f_2 + \frac{1}{64}A^4 f_3\right\}\right\}. \quad (47)$$

Thus, the second iterated complete solution of the oscillator (31) is

$$\theta_2(t) = \frac{24\Omega_1^2 A + 3\lambda_1 + \lambda_2}{24\Omega_1^2}\cos\theta - \frac{\lambda_1}{8\Omega_1^2}\cos 3\theta - \frac{\lambda_2}{24\Omega_1^2}\cos\theta. \quad (48)$$

6. Results and Discussion

To obtain approximate solutions of cube-root truly nonlinear oscillator, inverse cube-root truly nonlinear oscillator, and pendulum equation, we have applied an extended iteration method. Here, we have calculated the first, second, third, and fourth approximate frequencies of both oscillators and the first and second approximate frequencies of pendulum equation. All the results are given in Tables 1 and 2 and the frequency-amplitude relationships are given in Tables 3–5. To show the validity of the obtained solutions, we have compared these with the existing results determined by Mickens iteration method [25], Mickens HB method [25], He's amplitude-frequency formulation [34], modified He's homotopy perturbation method [15], improved harmonic balance method [6] and homotopy perturbation method [9] in Table 1, He's energy balanced method [21] in Table 3, Mickens iteration method [25], Mickens HB method [25] in Table 2, and homotopy perturbation method [13] in Table 5. The comparison between the third-order approximate solution of equation (8) for $A = 10$ and $A = 100$ together with the corresponding exact solution are presented in Figures 2 and 3. The comparison between the third-order approximate solution of equation (28) for $A = 1$ together with the corresponding exact solution are presented in Figure 4 and the comparison between the second-order approximate solution of equation (31) for $A = 1$ and $K = 1$ together with the corresponding exact solution are presented in Figure 5.

Here, the method of Mickens' iteration [25] is divergent, as error pattern shows an increasing phenomenon for corresponding iteration steps. The process of Mickens' HB method [13] is too much complicated for higher order approximations for its algebraic complexity, that is why the process was terminated up to second approximation and He could not determine the better solutions. Not only that, there is no flexibility to take any ordered harmonic terms in each iterated step. The improvement of solutions depends on only starting consideration. It is a disadvantage of this method.

For both of the oscillators, we can say, the adopted method is convergent because the solution yields lesser error in each forward iterative step:

(1) For cube-root TNL oscillator $\ddot{x} + x^{1/3} = 0$,

Table 1: Comparison of the approximate frequencies with the exact frequency Ω_{CRe} of $\ddot{x} + x^{1/3} = 0$.

$\Omega_{exact}(A) = \Omega_{CRe}(A) = (1.070451/A^{1/3})$				
Amplitude A	Ω_{CR0} Error (%)	Ω_{CR1} Error (%)	Ω_{CR2} Error (%)	Ω_{CR3} Error (%)
Mickens iteration method [25]	$(1.0491151/A^{1/3})$ 2.0	$(1.041424/A^{1/3})$ 2.7	—	—
Mickens harmonic balance method [25]	$(1.049115/A^{1/3})$ 2.0	$(1.063410/A^{1/3})$ 0.7	—	—
Zhang using He's amplitude-frequency formulation [17]	$(1.04912/A^{1/3})$ 2.0			—
Beléndez et al. using modified He's homotopy perturbation method [15]	$(1.07685/A^{1/3})$ 0.6	$(1.06922/A^{1/3})$ 0.17	$(1.07078/A^{1/3})$ 0.024	—
Lim and Wu using improved harmonic balance method [6]	$(1.07685/A^{1/3})$ 0.6	$(1.06928/A^{1/3})$ 0.17		—
Beléndez using homotopy perturbation method [9]	$(1.07685/A^{1/3})$ 0.6	$(1.06991/A^{1/3})$ 0.05		
Adopted method	$(1.076847/A^{1/3})$ 0.5975	$(1.070452/A^{1/3})$ 0.00	$(1.070895/A^{1/3})$ 0.04	$(1.070378/A^{1/3})$ 0.00

Table 2: Comparison of the approximate frequencies with exact frequency Ω_{ICRe} of $\ddot{x} + x^{-(1/3)} = 0$.

$\Omega_{exact}(A) = \Omega_{ICRe}(A) = (1.154700/A^{2/3})$				
Amplitude A	Ω_{ICR0} Error (%)	Ω_{ICR1} Error (%)	Ω_{ICR2} Error (%)	Ω_{ICR3} Error (%)
Mickens iteration method [25]	$(1.08148/A^{2/3})$ 6.3	$(1.07634/A^{2/3})$ 6.78	$(0.988591/A^{2/3})$ 14.38	—
Mickens HB method [25]	$(1.31329/A^{2/3})$ 13.7	$(1.18824/A^{2/3})$ 2.9	—	—
Adopted method	$(1.194298/A^{2/3})$ 3.43	$(1.146861/A^{2/3})$ 0.67	$(1.158141/A^{2/3})$ 0.29	$(1.156090/A^{2/3})$ 0.12

Table 3: Comparison of the approximate frequency-amplitude relations with exact and existing results of $\ddot{x} + x^{1/3} = 0$.

A	Ω_{CR0} Er (%)	Ω_{CR1} Er (%)	Ω_{CR2} Er (%)	Ω_{CR3} Er (%)	$\Omega_{[7]}^{HEBM}$ Er (%)	$\Omega_{[7]}^{ex}$
0.1	2.319992336 0.5973	2.306220896 0.0002	2.307130328 0.0396	2.307082081 0.0375	2.269958874 1.572	2.306216768
0.5	1.356739749 0.5973	1.348686162 0.0002	1.349218000 0.0396	1.349189785 0.0375	1.327480002 1.572	1.348683748
1	1.076845052 0.5973	1.070452916 0.0002	1.070875036 0.0396	1.070852642 0.0375	1.053621576 1.572	1.070451000
5	0.629742806 0.5973	0.626004663 0.0002	0.626251520 0.0396	0.626238424 0.0375	0.6161616353 1.572	0.626003542
10	0.499827197 0.5973	0.496860230 0.0002	0.497056161 0.0396	0.497045766 0.0375	0.4890478141 1.572	0.496859340
50	0.292300718 0.5973	0.290565625 0.0002	0.290680206 0.0396	0.290674127 0.0375	0.2859968965 1.572	0.290565105
100	0.231999233 0.5973	0.230622089 0.0002	0.230713032 0.0396	0.230708208 0.0375	0.2269958874 1.572	0.230621676
500	0.135673974 0.5973	0.134868616 0.0002	0.134921800 0.0396	0.134918978 0.0375	0.1327480002 1.572	0.134868374
1000	0.107684505 0.5973	0.107045291 0.0002	0.107087503 0.0396	0.107085264 0.0375	0.1053621576 1.572	0.107045100

Note. Here, $\Omega_{[7]}^{HEBM}$ represents the approximation frequency obtained by Ganji et al. [21].

$$\left|\Omega_{CR3} - \Omega_{CRe}\right| = |1.070378 - 1.070451| < \varepsilon. \quad (49)$$

(2) For inverse cube-root TNL oscillator $\ddot{x} + x^{-(1/3)} = 0$,

$$\left|\Omega_{IC3} - \Omega_{ICRe}\right| = |1.156090 - 1.154700| < \varepsilon. \quad (50)$$

(3) For pendulum equation $\ddot{\theta} + \Omega^2 \sin\theta = 0$,

Table 4: Comparison of the approximate frequency-amplitude relations with exact results of $\ddot{x} + x^{-1/3} = 0$.

A	Ω_{ICR0} Er (%)	Ω_{ICR0} Er (%)	Ω_{ICR0} Er (%)	Ω_{ICR0} Er (%)	Ω_{ex}
0.1	5.54344 3.4	5.32326 0.68	5.37561 0.29	5.36609 0.12	5.35964
0.5	1.89583 3.4	1.82053 0.68	1.83843 0.29	1.83518 0.12	1.83297
1	1.19430 3.4	1.14686 0.68	1.15814 0.29	1.15609 0.12	1.15470
5	0.40844 3.4	0.39222 0.68	0.39608 0.29	0.39538 0.12	0.39490
10	0.25730 3.4	0.24708 0.68	0.24951 0.29	0.24907 0.12	0.24877
50	0.08780 3.4	0.08450 0.68	0.08533 0.29	0.08518 0.12	0.08508
100	0.05543 3.4	0.05323 0.68	0.05376 0.29	0.05366 0.12	0.05360
500	0.01896 3.4	0.01821 0.68	0.01838 0.29	0.01835 0.12	0.01833
1000	0.01194 3.4	0.01147 0.68	0.01158 0.29	0.01156 0.12	0.01155

Table 5: Comparison of the approximate frequency-amplitude relations with exact and existing results of $\ddot{\theta} + K^2 \sin\theta = 0$.

A	K	$\Omega^{ex}_{[28]}$	Ω_0 Er (%)	Ω_1 Er (%)	$\Omega^{HPM}_{[28]}$ Er (%)
0.1	2	1.99875	1.99875 0.0000	1.99875 0.0000	1.99875 0.0000
0.5	4	3.937579	3.937665 0.0022	3.937583 0.0001	3.937665 0.0022
1	1	0.937792	0.938194 0.0429	0.937853 0.0065	0.938194 0.0429
1.5	1	0.860608	0.863202 0.3013	0.861781 0.136299	0.863202 0.3013
2	1	0.7525	0.763763 1.4968	0.747482 0.6668	0.763763 1.4968

Note. Here, $\Omega^{HPM}_{[28]}$ represents the approximation frequency obtained by Bayat et al. [13].

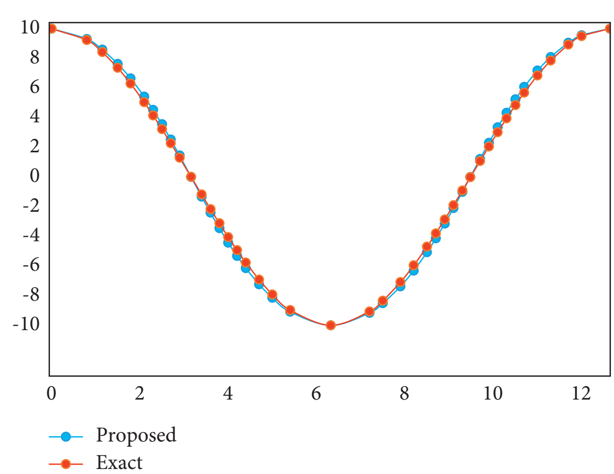

Figure 2: A comparison between the third-order approximate solutions of $\ddot{x} + x^{1/3} = 0$ for $A = 10$ together with the corresponding exact solution.

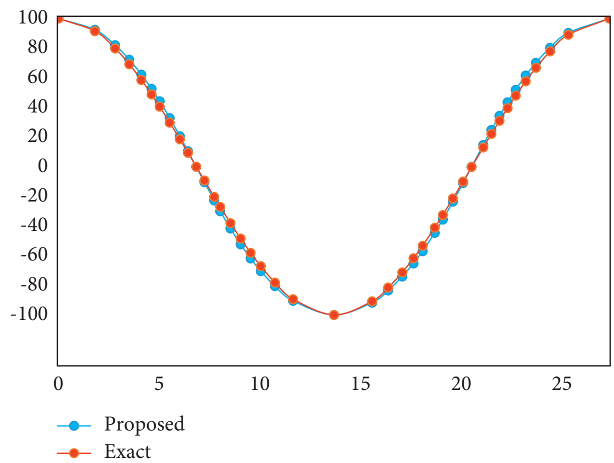

FIGURE 3: A comparison between the third-order approximate solutions of $\ddot{x} + x^{1/3} = 0$ for $A = 100$ together with the corresponding exact solution.

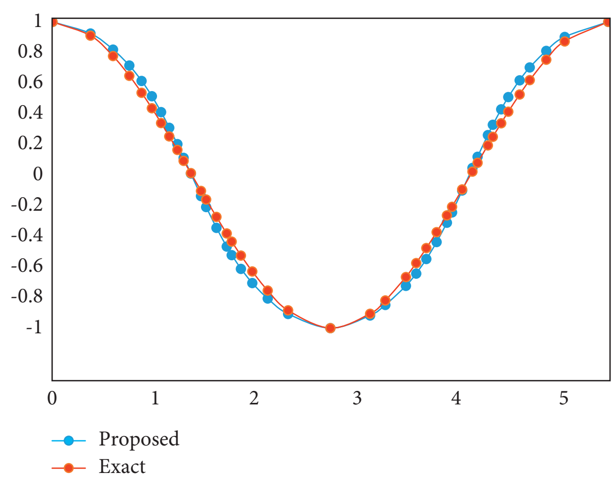

FIGURE 4: A comparison between the third-order approximate solutions of $\ddot{x} + x^{-(1/3)} = 0$ for $A = 1$ together with the corresponding exact solution.

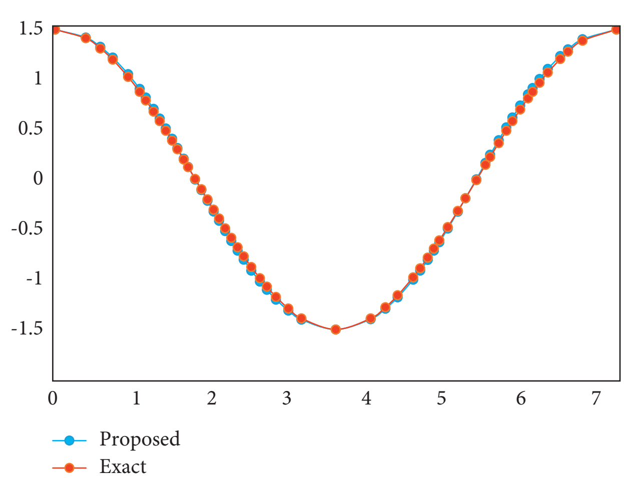

FIGURE 5: A comparison between the second-order approximate solutions of $\ddot{\theta} + K^2 \sin\theta = 0$ for $A = 1.5$ and $K = 1$ together with the corresponding exact solution.

$$\left|\omega_1 - \omega_{ex}\right| = |0.937853 - 0.937792| < \varepsilon, \quad \text{for } a = 1, \Omega = 1, \tag{51}$$

(i) where ε is taken to be of very small positive quantity.
(ii) Also, the consistency of the adopted method is acquired because

(4) For cube-root TNL oscillator $\ddot{x} + x^{1/3} = 0$,

$$\left|\Omega_{CR3} - \Omega_{CRe}\right| = |1.070378 - 1.070451| \approx 0. \tag{52}$$

(5) For inverse cube-root TNL oscillator $\ddot{x} + x^{-(1/3)} = 0$,

$$\left|\Omega_{ICR3} - \Omega_{ICRe}\right| = |1.156090 - 1.154700| \approx 0. \tag{53}$$

(6) For pendulum equation $\ddot{\theta} + \Omega^2 \sin\theta = 0$,

$$\left|\omega_1 - \omega_{ex}\right| = |0.937853 - 0.937792| \approx 0, \quad \text{for } a = 1, \Omega = 1. \tag{54}$$

7. Conclusion

After reflecting on each point of view about all the methods compared in Tables 1–3 and 5, we conclude that the adopted method is significantly better than each comparable stages which have been shown by other methods. Finally, we summarized:

(i) The proposed method is a powerful technique for analyzing random oscillations. The technique is also easy and useful to obtain approximate frequencies and the corresponding periodic solutions of strongly nonlinear oscillator.
(ii) The proposed method gives high validity for both small and large initial amplitudes of oscillations and better results of the approximate frequencies and corresponding periodic solutions than those obtained by other existing results.
(iii) The percentage errors for the fourth approximate frequencies of cube-root truly nonlinear oscillator and inverse cube-root truly nonlinear oscillator are 0.006 and 0.12, respectively. Also, the percentage error for the second approximate frequency (for amplitude = 1) of equation of pendulum is 0.0065.
(iv) It is investigated that most of the researchers have used the procedure to modify the method to improve the solutions in the iteration method, but we have paid attention to rearranging the leading oscillators with their own merit and choosing suitable harmonic terms from trigonometric expansion. It has been concluded that these two are also imperative matter for obtaining improved solutions.
(v) The proposed method is also convergent for the considered equations.

Nomenclature

A: Initial oscillation amplitude
x: Dimensionless displacement (m)
$f_1, f_2, f_3, \lambda_1, \lambda_2, k, K$: Constant parameters
t: Time (s)
G, H, f: General nonlinear functions
g: Gravitational acceleration (m/s^2)
l: Length of the pendulum (m)
$x_{CR0}, x_{CR1}, x_{CR2}$: First, second, and third approximate analytic solutions of the cube-root TNL oscillator, respectively
$x_{ICR0}, x_{ICR1}, x_{ICR2}$: First, second, and third approximate analytic solutions of the inverse cube-root TNL oscillator, respectively
x_0, x_1: First and second approximate analytic solutions of the pendulum equation, respectively
Er (%): Percentage error.

Greek Letters

θ: Angle from the vertical to the pendulum (°)
$\Omega_{CR0}, \Omega_{CR1}, \Omega_{CR2}, \Omega_{CR3}$: First, second, third, and fourth approximate frequencies of the cube-root TNL oscillator, respectively
$\Omega_{ICR0}, \Omega_{ICR1}, \Omega_{ICR2}, \Omega_{ICR3}$: First, second, third, and fourth approximate frequencies of the inverse cube-root TNL oscillator, respectively
Ω_0, Ω_1: First and second approximate frequencies of the pendulum equation, respectively
ε: Very small positive quantity
Ω_{ex}: Exact frequency.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Both authors have equal contributions. They have read and approved the final version of the paper.

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