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An improved Mickens' solution for nonlinear vibrations

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ABSTRACT

The Duffing equation and other nonlinear equations of motion of nonlinear vibrations are widely applied in the fields of engineering and science, especially mechanical and electrical engineering. The modified Mickens extended iteration method has been utilized in this article to investigate for further accurate solutions to the Duffing equation and an equation of motion of nonlinear vibrations. The third approximate frequency shows good agreement with the exact result for both oscillators. For simplicity in algebraic calculations, the increasing harmonic terms have been taken to the fifth order. In order to establish the reliability and effectiveness of the proposed approach, the results obtained through this method are compared with those available in the literature. The comparison of the obtained solutions with numerical solutions shows a remarkable accuracy. Also, it becomes evident that the modified Mickens extended iteration method is significantly simpler, more accurate, efficient, and straightforward than other existing methods. The highest percentage error in the third approximate frequency of an equation of motion of nonlinear vibrations is less than **0.0488**, and the maximal percentage error in the third approximate frequency of the Duffing oscillator is less than **0.0065**. The proposed technique is principally exhibited in nonlinear models where the nonlinear terms are strong, but it can also be broadly applicable to other problems arising in engineering.

Mathematics Subject Codes: 34A34, 34C25, 37N15 (AMS Mathematics Subject Classification 2010)

1. Introduction

Nonlinear systems are extensively employed across various fields, including science, engineering, and technology, among others. The behavior of complex nonlinear dynamical systems is described by a class of mathematical models. These systems exhibit a feature in which they are extremely sensitive to their initial conditions, suggesting that even slight modifications to these initial conditions can lead to significantly distinct outcomes. The inherent complexity and sensitivity of these systems make controlling them a formidable challenge. Therefore, researchers and scientists established and extended several iterative methods, as for instance the perturbation method [29,30], He's homotopy perturbation method [9], harmonic balance method [10,22,24,25], Haque's iterative method [17,18], Mickens' iterative method [26–28], extended iterative method [1,2,19,20], He's max-min method [5], rational energy balance method [11], energy balance method [3,21], and He's energy balance method [14], etc. One of them, the perturbation method, is well-known and well-established and takes into account small terms, but there are many engineering models where small parameters cannot be found, and even when they are found, the accuracy rate of the approximations obtained by the perturbation method is low. Another known approach, the homotopy perturbation method for nonlinear problems has been extensively studied by scientists and engineers because it continuously converts a difficult problem into a straightforward, easy and solvable one. With homotopy perturbation method, a broad range of nonlinear problems can be resolved using approximations that swiftly converge to accurate solutions. For the equations involving both weakly and strongly nonlinear terms, this method is effective and feasible. But the solution processes of the approach are difficult and are suitable for first-order solutions. Another well-known technique for the handling large nonlinearities is the harmonic balance method. However, the use of the approach introduces a number of complicated terms. One of the well-established technique for solving nonlinear equations is the parameterized perturbation method. This method, which was previously hardly ever used, is a potent instrument for handling weakly nonlinear issues; nonetheless, it has a number of drawbacks for strongly nonlinear problems. But the proposed

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Nomenclature

A	Amplitude of the Initial Oscillation
x	Displacement without Dimension (m)
$f, g, h, f_{M_0}, f_{M_1}, f_{M_2}, f_{M_{31}}, f_{M_{32}}, f_{M_{33}}, f_{C_{11}}, f_{C_{12}}, f_{C_{13}}, f_{C_{21}}, f_{C_{22}}, f_{C_{23}}, f_{C_{31}}, f_{C_{32}}, f_{C_{33}}, f_{D_{11}}, f_{D_{12}}, f_{D_{13}}, f_{D_{21}}, f_{D_{22}}, f_{D_{23}}, f_{D_{31}}, f_{D_{32}}, f_{D_{33}}, f_{MD}, g_{MD}, f_{MD_{31}}, f_{MD_{32}}, f_{MD_{33}}$	Constant Parameters
t	Time (S)
G,H,F	Nonlinear Functions
$x_{C_0}, x_{C_1}, x_{C_2}$	First, second, and third periodic solutions of an equation of motion of nonlinear vibrations, respectively.
$x_{M_0}, x_{M_1}, x_{M_2}$	First, second, and third periodic solutions of an equation of motion of nonlinear vibrations, respectively obtained by Mickens' iteration method.
$x_{D_0}, x_{D_1}, x_{D_2}$	The Duffing oscillator's first, second, and third periodic solutions, respectively.
$x_{MD_0}, x_{MD_1}, x_{MD_2}$	The Duffing oscillator's first, second, and third periodic solutions, respectively obtained by Mickens' iteration method.
Error (%)	Percentage Error

Greek letters

α, β, λ	Constant Parameters
Ω	Circular frequency of the free un-damped vibrations.
$\Omega_{C_0}, \Omega_{C_1}, \Omega_{C_2}$	First, second, and third approximate frequencies of an equation of motion of nonlinear vibrations, respectively.
$\Omega_{M_0}, \Omega_{M_1}, \Omega_{M_2}$	First, second, and third approximate frequencies of an equation of motion of nonlinear vibrations, respectively obtained by Mickens' iteration method.
$\Omega_{D_0}, \Omega_{D_1}, \Omega_{D_2}$	The Duffing oscillator's first, second, and third approximate frequencies, respectively.
$\Omega_{MD_0}, \Omega_{MD_1}, \Omega_{MD_2}$	The Duffing oscillator's first, second, and third approximate frequencies, respectively obtained by Mickens' iteration method.
θ	New variable equal to circular frequency× time
Ω_{ex}	Exact frequency

method is the most efficient method that is appropriate for both small and strong nonlinearities.

For strong nonlinearities, the Duffing equation and nonlinear equations of motion of nonlinear vibrations are the most important. Engineering, science, medicine, and other real-fields have benefited from the research on the Duffing equation and nonlinear equations of motion of nonlinear vibrations with strong nonlinearities. Due to its importance, there is a considerable demand among researchers for the Duffing equation and nonlinear equations of motion of nonlinear vibrations with strong nonlinearities. Although research on nonlinear systems in the presence of third- and fifth-power nonlinear factors is challenging and sensitive, researchers have conducted research on nonlinear vibrations or the cubic-quintic Duffing oscillator using a variety of techniques. For example, Sedighi et al. used the stiffness analytical approaches [33], the homotopy perturbation method with an auxiliary term [33], the max-min approach [33], and the iteration perturbation method [33], Younesian et al. used the He's frequency-amplitude formulation [34], the He's energy balance method [34], Beléndez et al. used the arithmetic-geometric mean [8], J-H He used a one-step frequency formulation [16], Azimi et al. used the modified homotopy perturbation method [6], the energy balance method [6], Ganji et al. used the iteration perturbation method [14,15], Pirbodaghi et al. used the homotopy analysis method [31], and Razzak used a combined of homotopy perturbation method and variational formulation [32]. They only solve for the first order, which offers good accuracy, but when they solve for the higher order, the solutions contain a lot of complexity. Lai et al. used the Newton–harmonic balance approach [24], Khan et al. used the coupled homotopy-variational formulation [23], and Chowdhury et al. used the harmonic balance method with the truncation property [10], the harmonic balance method without the truncation property [10] solved higher-order. But, compared to them, the outcomes applying the proposed method are much better. Although Zuniga et al. [36], Salas et al. [42,44], and Belendez et al. [7,8] have found exact solutions to the equations of motion of nonlinear vibrations, the solution procedures of those studies are complicated. Additionally, the solutions of the oscillator of those studies hold multifaceted algebraic equations, including Jacobian functions, which are tricky to solve and time-consuming. The Duffing oscillator has also been studied by a number of researchers, including Daeichin et al. [11], Younesian et al. [34,35], Durmaz et al. [12,13], Askari et al. [4], Chen & Liu [41], Salas et al. [43,45], and Hosen et al. [3], based on various methodologies, but the proposed technique outperforms them in terms of accuracy.

Mickens [28] established an "extended iteration" technique. The technique was then improved by Lim & Wu [40] and Hu [39]. Later, Haque & Hossain [20] and Hossain & Haque [37] worked on the technique and significantly modified it.

The proposed method stands out due to its capacity to yield high-accuracy results regardless of whether the initial amplitudes are small or large [1,2,20,37]. On the other hand, most of the approaches give high-accuracy results for smaller initial amplitudes and low accuracy for large ones. Furthermore, the proposed method yields outstanding results for higher-order solutions, whereas most of the methods perform adequately for first-order solutions but falter when it comes to higher-order ones. The proposed method also provides a suitable way for solving a number of complicated nonlinear models and enhances the measure of validity of approximate solutions as well as the frequencies of the strongly nonlinear models.

In this paper, Mickens' extended iteration scheme has been used, but the solution procedure and rearranging of nonlinear terms are different. It has been applied with some truncated Fourier series and different harmonic terms for different iteration steps to improve the solution. The proposed method has been applied to solve the Duffing equation and an equation of motion of nonlinear vibrations. After applying the proposed method, more accurate, feasible, and compatible outcomes have been found. From the modification, effective simplification has been found so that there are no algebraic complicities in the solution. The computed outcomes obtained using the proposed approach have been compared to the results of other existing methods and to the exact result. The obtained outcomes rapidly converge to the exact results, and errors are much smaller than those existing in the literature.

2. The methodology

The modified extended iterative method works in four steps:

(i) Consider a governing equation;
(ii) Convert it into standard form;
(iii) Consider iterative scheme; and finally
(iv) Formulate the extended iteration scheme.

First: consider a generalized single degree of freedom nonlinear equation of motion in the following form:

$$\frac{d^2x}{dt^2} + F(x) = 0 \Rightarrow \ddot{x} + F(x) = 0, \tag{1}$$

using the initial conditions $x(0) = A,\ \dot{x}(0) = 0$, and $\ddot{x}$ represents the double differentiation of x with respect to t time.

Standard form: It is possible to rewrite Eq. (1) as

$$\ddot{x} + \Omega^2 x = \Omega^2 x - F(x) = G(x, \Omega), \tag{2}$$

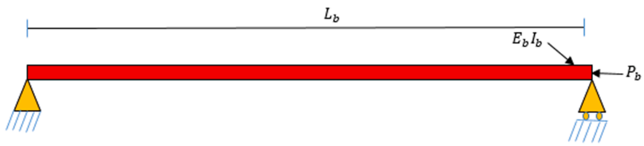

Fig. 1. A uniform simply supported nonlinear vibrations.

where Ω represents circular frequency of the free un-damped vibrations.

Iterative scheme: The direct iteration technique of Eq. (2) has been taken into consideration as

$$\ddot{x}_{k+1} + \Omega_k^2 x_{k+1} = G(x_k,\ \Omega_k); \quad k = 0,\ 1,\ 2,\ \ldots \tag{3}$$

using initial supposition

$$x_0(t) = A\cos(\Omega_0\, t), \tag{4}$$

and

$$x_{k+1}(0) = A, \quad \dot{x}_{k+1}(0) = 0 \tag{5}$$

where A indicates the amplitude.

Extended Iteration Scheme: We have considered the Extended Iteration Scheme of Eq. (2) as

$$\ddot{x}_{k+1} + \Omega_k^2 x_{k+1} = G(x_k, x_k) + G_x(x_0, \Omega_k)(x_k - x_0) \tag{6}$$

where $G_x = \frac{\partial G}{\partial x}$ and x_{k+1} satisfies the conditions (5). We have further considered the increasing harmonic terms, which are taken to be up to the fifth.

Here $x_k(t)$; $k = 1, 2, 3, \ldots$and Ω_k ; $k = 0, 1, 2, \ldots$are the first, second, third,…approximate roots respectively, and corresponding frequencies of the oscillators, respectively, obtained by avoiding the secular terms in each step.

3. 3A. Solution procedure: nonlinear vibrations

Physical model of a nonlinear vibrations

where L_b indicates the length of a nonlinear vibrations, I_b indicates the moment of inertia, m_b indicates the mass proportional to length, and E_b is the elasticity's modulus lying on an elastic base that is axially squeezed by a loading P_b, where P_b is constant, as depicted in Fig. 1. Indicating by W_b represents the transverse deflection.

Mathematical model of a nonlinear vibrations:

The mathematical model of a nonlinear vibrations equation derived by [23] can be expressed as follows:

$$\ddot{x} + \alpha\, x + \beta\, x^3 + \lambda\, x^5 = 0,$$

with the initial condition

$$x(0) = A, \dot{x}(0) = 0. \tag{7}$$

where

$$\alpha = \pi^4 - \frac{P_b L_b \pi^2}{E_b I_b}, \beta = -\frac{3}{8}\pi^6 - \frac{3}{8}\frac{P_b L_b^2 \pi^4}{E_b I_b}, \lambda = \frac{15}{64}\pi^8. \tag{8}$$

The Eq. (7) also known as cubic-quintic Duffing oscillator which is stated by [6,14–16,24,34], where α, β and λ are real system constant parameters. Changing the values ofα, β and λ causing the behavior of the solution to change.

Solution Procedure of a nonlinear vibrations

Mickens Iteration Method

The Eq. (7) is written as

$$\ddot{x}_M + {\Omega_M}^2 x_M = {\Omega_M}^2 x_M - \alpha\, x_M - \beta\, {x_M}^3 - \lambda\, {x_M}^5 = G(x_M,\ \Omega_M) \tag{9}$$

where

$$G(x_M,\ \Omega_M) = {\Omega_M}^2 x - \alpha\, x_M - \beta\, {x_M}^3 - \lambda\, {x_M}^5 \tag{10}$$

Here x_M and Ω_M denotes the solutions and the frequencies obtained by Mickens.

Applying Mickens' iteration method defined in Eq. (3) to Eq. (9), we found

$$\ddot{x}_{M_{K+1}} + \Omega_{M\ K}^2 x_{M_{K+1}} = \left(\Omega_{M_K}^2 x_{M_K} - \alpha\, x_{M_K} - \beta\, x_{M_K}^3 - \lambda\, x_{M_K}^5\right) \tag{11}$$

For the first iteration, $k = 0$, we attain

$$\ddot{x}_{M_0} + \Omega_{M_0}^2 x_{M\ 1} = \Omega_{M_0}^2 A\cos\theta - \alpha\, A\cos\theta - \beta\,(A\cos\theta)^3 - \lambda\,(A\cos\theta)^5 \tag{12}$$

where $\theta = \Omega_M t$, and $\Omega_M = \Omega$

Applying Fourier series in the right side of Eq. (12), it is found

$$\ddot{x}_{M\ 1} + \Omega_{M\ 0}^2 x_{M\ 1} = \left(\Omega_{M\ 0}^2 A - \alpha A - \frac{3}{4}\beta A^3 - \frac{5}{8}\lambda A^5\right)\cos\theta - \left(\frac{1}{4}\beta A^3 + \frac{5}{16}\lambda A^5\right)\cos 3\theta - \frac{1}{16}\lambda A^5\cos 5\theta. \tag{13}$$

By avoiding the secular terms, taking $\cos\theta = 0$. Then we have

$$\Omega_{M_0} = \sqrt{\alpha + \frac{3}{4}\beta A^2 + \frac{5}{8}\lambda A^4}. \tag{14}$$

The first approximate frequency of the Eq. (7) is shown in Eq. (14).

After being simplified, the Eq. (13) gives the first approximate solution of the Eq. (7) is

$$x_{M_1}(t) = f\cos\theta + g\cos 3\theta + h\cos 5\theta, \tag{15}$$

where

$$f = \frac{A(-3A^2\beta - 4A^4\lambda + 96\Omega^2)}{96\Omega^2},$$

$$g = \frac{A^3(4\beta + 5A^2\lambda)}{128\Omega^2},$$

$$h = \frac{A^5\lambda}{384\Omega^2}, \Omega^2 = \Omega_{M_0}^2. \tag{16}$$

Applying the Mickens' iteration method for the second order solution, we derive

$$\ddot{x}_{M_2} + \Omega_{M_1}^2 x_{M_2} = f_{M_1}\cos\theta + f_{M_2}\cos 3\theta + f_{M_3}\cos 5\theta, \tag{17}$$

where

$$\begin{aligned} f_{M_1} = & -A\alpha - \frac{3A^3\beta}{4} - \frac{5A^5\lambda}{8} + \frac{45A^{15}\beta^5\lambda}{536870912\Omega^{10}} + \frac{875A^{17}\beta^4\lambda^2}{1610612736\Omega^{10}} \\ & + \frac{27265A^{19}\beta^3\lambda^3}{19327352832\Omega^{10}} + \frac{35455A^{21}\beta^2\lambda^4}{19327352832\Omega^{10}} + \frac{6649685A^{23}\beta\lambda^5}{5566277615616\Omega^{10}} \\ & + \frac{5204815A^{25}\lambda^6}{1669883284684 8\Omega^{10}} - \frac{25A^{13}\beta^4\lambda}{4194304\Omega^8} - \frac{525A^{15}\beta^3\lambda^2}{16777216\Omega^8} - \frac{12425A^{17}\beta^2\lambda^3}{201326592\Omega^8} \\ & - \frac{196385A^{19}\beta\lambda^4}{3623878656\Omega^8} - \frac{518245A^{21}\lambda^5}{28991029248\Omega^8} + \frac{3A^9\beta^4}{65536\Omega^6} + \frac{97A^{11}\beta^3\lambda}{262144\Omega^6} \\ & + \frac{523A^{13}\beta^2\lambda^2}{524288\Omega^6} + \frac{28231A^{15}\beta\lambda^3}{25165824\Omega^6} + \frac{12875A^{17}\lambda^4}{28311552\Omega^6} - \frac{9A^7\beta^3}{4096\Omega^4} - \frac{39A^9\beta^2\lambda}{4096\Omega^4} \\ & - \frac{455A^{11}\beta\lambda^2}{32768\Omega^4} - \frac{2005A^{13}\lambda^3}{294912\Omega^4} + \frac{A^3\alpha\beta}{32\Omega^2} + \frac{3A^5\beta^2}{64\Omega^2} + \frac{A^5\alpha\lambda}{24\Omega^2} + \frac{29A^7\beta\lambda}{256\Omega^2} \\ & + \frac{35A^9\lambda^2}{512\Omega^2} + A\Omega 1^2 - \frac{A^3\beta\Omega 1^2}{32\Omega^2} - \frac{A^5\lambda\Omega 1^2}{24\Omega^2}, \end{aligned} \tag{18}$$

$$f_{M_2} = -\frac{A^3\beta}{4}-\frac{5A^5\lambda}{16}-\frac{65A^{15}\beta^5\lambda}{536870912\Omega^{10}}-\frac{2485A^{17}\beta^4\lambda^2}{3221225472\Omega^{10}}-\frac{38045A^{19}\beta^3\lambda^3}{19327352832\Omega^{10}}-\frac{583135A^{21}\beta^2\lambda^4}{231928233984\Omega^{10}}-\frac{8948345A^{23}\beta\lambda^5}{5566277615616\Omega^{10}}-\frac{13747375A^{25}\lambda^6}{33397665693696\Omega^{10}}$$
$$+\frac{125A^{13}\beta^4\lambda}{16777216\Omega^8}+\frac{1925A^{15}\beta^3\lambda^2}{50331648\Omega^8}+\frac{1855A^{17}\beta^2\lambda^3}{25165824\Omega^8}+\frac{229075A^{19}\beta\lambda^4}{3623878656\Omega^8}+\frac{3540215A^{21}\lambda^5}{173946175488\Omega^8}-\frac{A^9\beta^4}{16384\Omega^6}-\frac{111A^{11}\beta^3\lambda}{262144\Omega^6}-\frac{2171A^{13}\beta^2\lambda^2}{2097152\Omega^6}-\frac{244937A^{15}\beta\lambda^3}{226492416\Omega^6}$$
$$-\frac{93385A^{17}\lambda^4}{226492416\Omega^6}+\frac{9A^7\beta^3}{4096\Omega^4}+\frac{33A^9\beta^2\lambda}{4096\Omega^4}+\frac{973A^{11}\beta\lambda^2}{98304\Omega^4}+\frac{4805A^{13}\lambda^3}{1179648\Omega^4}-\frac{A^3\alpha\beta}{32\Omega^2}-\frac{3A^5\beta^2}{128\Omega^2}-\frac{5A^5\alpha\lambda}{128\Omega^2}-\frac{5A^7\beta\lambda}{128\Omega^2}-\frac{35A^9\lambda^2}{3072\Omega^2}+\frac{A^3\beta\Omega1^2}{32\Omega^2}+\frac{5A^5\lambda\Omega1^2}{128\Omega^2}, \tag{19}$$

$$f_{M_3} = -\frac{A^5\lambda}{16}+\frac{A^{15}\beta^5\lambda}{536870912\Omega^{10}}-\frac{35A^{17}\beta^4\lambda^2}{3221225472\Omega^{10}}-\frac{35A^{19}\beta^3\lambda^3}{402653184\Omega^{10}}-\frac{43645A^{21}\beta^2\lambda^4}{231928233984\Omega^{10}}-\frac{949915A^{23}\beta\lambda^5}{5566277615616\Omega^{10}}-\frac{632827A^{25}\lambda^6}{11132555231232\Omega^{10}}+\frac{25A^{13}\beta^4\lambda}{16777216\Omega^8}$$
$$+\frac{455A^{15}\beta^3\lambda^2}{50331648\Omega^8}+\frac{1015A^{17}\beta^2\lambda^3}{50331648\Omega^8}+\frac{2975A^{19}\beta\lambda^4}{150994944\Omega^8}+\frac{1241555A^{21}\lambda^5}{173946175488\Omega^8}-\frac{55A^{11}\beta^3\lambda}{524288\Omega^6}-\frac{2705A^{13}\beta^2\lambda^2}{6291456\Omega^6}-\frac{14791A^{15}\beta\lambda^3}{25165824\Omega^6}-\frac{60695A^{17}\lambda^4}{226492416\Omega^6}+\frac{3A^7\beta^3}{4096\Omega^4}+\frac{19A^9\beta^2\lambda}{4096\Omega^4}$$
$$+\frac{553A^{11}\beta\lambda^2}{65536\Omega^4}+\frac{5605A^{13}\lambda^3}{1179648\Omega^4}-\frac{3A^5\beta^2}{128\Omega^2}-\frac{A^5\alpha\lambda}{384\Omega^2}-\frac{A^7\beta\lambda}{16\Omega^2}-\frac{125A^9\lambda^2}{3072\Omega^2}+\frac{A^5\lambda\Omega1^2}{384\Omega^2}, \tag{20}$$

By avoiding the secular terms, taking $\cos\theta = 0$. Then, the second approximate frequency of the Eq. (7) has been obtained using Eqs. (17) and (18).

$$\Omega_{M_1} = \frac{1}{4608\sqrt{2}}\Bigg(\sqrt{(}\Big(15925248(4\alpha+3A^2\beta)^2(32\alpha+23A^2\beta)(2048\alpha^3+4608A^2\alpha^2\beta+3408A^4\alpha\beta^2+831A^6\beta^3)+10368A^4\Big(5972688896\alpha^5$$
$$+22139633664A^2\alpha^4\beta+32743489536A^4\alpha^3\beta^2+24154908672A^6\alpha^2\beta^3+8889293184A^8\alpha\beta^4+1305721881A^{10}\beta^5\Big)\lambda+10368A^8\Big(9159311360\alpha^4$$
$$+27052032000A^2\alpha^3\beta+29893069824A^4\alpha^2\beta^2+14649091296A^6\alpha\beta^3+2686455661A^8\beta^4\Big)\lambda^2+864A^{12}\Big(89312788480\alpha^3$$
$$+197129571072A^2\alpha^2\beta+144729405408A^4\alpha\beta^2+35348428103A^6\beta^3\Big)\lambda^3+1440A^{16}\Big(24354826240\alpha^2+35721035536A^2\alpha\beta$$
$$+13072578999A^4\beta^2\Big)\lambda^4+15A^{20}\Big(563870755008\alpha+412290970559A^2\beta\Big)\lambda^5+846077683385A^{24}\lambda^6\Big)\Big/\Big((8\alpha+6A^2\beta+5A^4\lambda)^4$$
$$\Big(96\alpha+69A^2\beta+56A^4\lambda\Big)\Big)\Big)\Bigg). \tag{21}$$

where $\Omega_{M_1} = \Omega1 =$ second approximate frequency

After being simplified, the second approximate solution of the Eq. (7) is

$$x_{M_2}(t) = f_{M_{31}}\cos\theta + f_{M_{33}}\cos 3\theta + f_{M_{35}}\cos 5\theta, \tag{22}$$

where

$$f_{M_{31}} = -\Big(\Big(A\big(1547044380570A^{22}\beta\lambda^5+399402810835A^{24}\lambda^6-802345520625352704 0\Omega1^{12}+2080A^{20}\lambda^4(1153937247\beta^2$$
$$-10091779792\lambda\Omega1^2)+137280A^{18}\beta\lambda^3(13581237\beta^2-469913648\lambda\Omega1^2)+255058771968A^{10}\beta\lambda\Omega1^4(1689\beta^2-47600\lambda\Omega1^2)$$
$$-18218483712A^{12}\lambda\Omega1^2(405\beta^4-60392\beta^2\lambda\Omega1^2+298080\lambda^2\Omega1^4)+7413120A^{14}\beta\lambda(15237\beta^4-5159328\beta^2\lambda\Omega1^2+160328000\lambda^2\Omega1^4)$$
$$+137280A^{16}\lambda^2(5281965\beta^4-542346912\beta^2\lambda\Omega1^2+3394892672\lambda^2\Omega1^4)-775378666782720A^6(3\beta^3\Omega1^6-80\beta\lambda\Omega1^8)$$
$$+340078362624A^8(171\beta^4\Omega1^4-26904\beta^2\lambda\Omega1^6+81440\lambda^2\Omega1^8)+31341621899427840A^2\beta\Omega1^8(\alpha+8\Omega1^2-\Omega2^2)$$
$$+1741201216634880A^4\Omega1^8(18\beta^2+\lambda(23\alpha+192\Omega1^2-23\Omega2^2))\big)\Big)\Big/(8023455206253527040\Omega1^{12})\Big), \tag{23}$$

$$f_{M_{33}} = \frac{1}{267181325549568\Omega1^{12}}A^3\Big(53690070A^{20}\beta\lambda^5+13747375A^{22}\lambda^6+1680A^{18}\lambda^4\Big(49983\beta^2-404596\lambda\Omega1^2\Big)+20160A^{16}\beta\lambda^3\Big(3261\beta^2$$
$$-104720\lambda\Omega1^2\Big)+42467328A^8\beta\lambda\Omega1^4\Big(333\beta^2-7784\lambda\Omega1^2\Big)-221184A^{10}\lambda\Omega1^2\Big(1125\beta^4-156312\beta^2\lambda\Omega1^2+615040\lambda^2\Omega1^4\Big)$$
$$+5760A^{14}\lambda^2\Big(4473\beta^4-427392\beta^2\lambda\Omega1^2+2390656\lambda^2\Omega1^4\Big)+2304A^{12}\beta\lambda\Big(1755\beta^4-554400\beta^2\lambda\Omega1^2+15675968\lambda^2\Omega1^4\Big)$$
$$-8153726976A^4\Big(9\beta^3\Omega1^6-160\beta\lambda\Omega1^8\Big)+679477248A^6\Big(3\beta^4\Omega1^4-396\beta^2\lambda\Omega1^6+560\lambda^2\Omega1^8\Big)+1043677052928\beta\Omega1^8\Big(\alpha+8\Omega1^2$$
$$-\Omega2^2\Big)+260919263232A^2\Omega1^8\Big(3\beta^2+5\lambda\Big(\alpha+8\Omega1^2-\Omega2^2\Big)\Big)\Big), \tag{24}$$

$$
\begin{aligned}
f_{M_{35}} = & \frac{1}{267181325549568\Omega 1^{12}} A^5 \Big(1899830A^{18}\beta\lambda^5 + 632827A^{20}\lambda^6 \\
& + 560A^{16}\lambda^4 \left(3741\beta^2 - 141892\lambda\Omega 1^2 \right) + 21233664A^6\beta\lambda\Omega 1^4 \Big(55\beta^2 \\
& - 4424\lambda\Omega 1^2 \Big) + 322560A^{14}\beta\lambda^3 \left(3\beta^2 - 680\lambda\Omega 1^2 \right) \\
& + 905969664A^4\lambda\Omega 1^6 \left(-57\beta^2 + 500\lambda\Omega 1^2 \right) \\
& - 6912A^{10}\beta\lambda \left(3\beta^4 + 14560\beta^2\lambda\Omega 1^2 - 946624\lambda^2\Omega 1^4 \right) \\
& - 368640A^8\lambda\Omega 1^2 \Big(45\beta^4 \\
& - 12984\beta^2\lambda\Omega 1^2 + 143488\lambda^2\Omega 1^4 \Big) + 1920A^{12}\lambda^2 \Big(63\beta^4 \\
& - 116928\beta^2\lambda\Omega 1^2 + 1553792\lambda^2\Omega 1^4 \Big) - 2717908992A^2 \Big(3\beta^3\Omega 1^6 \\
& - 256\beta\lambda\Omega 1^8 \Big) + 28991029248\Omega 1^8 \left(9\beta^2 + \lambda \left(\alpha + 24\Omega 1^2 - \Omega 2^2 \right) \right) \Big).
\end{aligned}
\tag{25}
$$

where, $\Omega 1 = \Omega_{M_1}$ and $\Omega 2 = \Omega_{M_2}$.

Applying the Mickens' iteration method and then after being simplified, we obtain the third approximate frequency,

$$
\begin{aligned}
\Omega_{M_2} = & \frac{1}{294912\sqrt{2}} \Big(\surd \Big(\Big(-5A^{14}\lambda \Big(279936\beta^5 + 1814400A^2\beta^4\lambda \\
& + 4711392A^4\beta^3\lambda^2 + 6126624A^6\beta^2\lambda^3 + 3989811A^8\beta\lambda^4 + 1040963A^{10}\lambda^5 \Big) \\
& + 2880A^{12}\lambda \Big(34560\beta^4 + 181440A^2\beta^3\lambda + 357840A^4\beta^2\lambda^2 + 314216A^6\beta\lambda^3 \\
& + 103649A^8\lambda^4 \Big) \Omega 1^2 - 73728A^8 \Big(10368\beta^4 + 83808A^2\beta^3\lambda \\
& + 225936A^4\beta^2\lambda^2 + 254079A^6\beta\lambda^3 + 103000A^8\lambda^4 \Big) \Omega 1^4 \\
& + 56623104A^6 \left(648\beta^3 + 2808A^2\beta^2\lambda + 4095A^4\beta\lambda^2 + 2005A^6\lambda^3 \right) \Omega 1^6 \\
& - 10871635968A^2 \Big(16\alpha \left(3\beta + 4A^2\lambda \right) + 3A^2 \left(4\beta + 5A^2\lambda \right) \Big(6\beta \\
& + 7A^2\lambda \Big) \Big) \Omega 1^8 + 2087354105856 (8\alpha + 6A^2\beta + 5A^4\lambda) \Omega 1^{10}) / (\Omega 1^8 (\\
& - 3A^2\beta - 4A^4\lambda + 96\Omega 1^2)) \Big) \Big).
\end{aligned}
\tag{26}
$$

Proposed Method

The proposed method, the modified Mickens extended iteration method, is a novel iterative technique that combines the Mickens extended iteration method and the Mickens iteration method to solve a wide range of complicated nonlinear systems. To enhance the approximate solution, the proposed method has been used with a few truncated Fourier series along with various harmonic terms for various iteration stages. After being modified, there has been a successful simplification so that there are no algebraic complications.

The Eq. (7) can be written as

$$\ddot{x}_c + \Omega_c{}^2 x_c = \Omega_c{}^2 x_c - \alpha\, x_c - \beta\, x_c{}^3 - \lambda\, x_c{}^5 = \; G(x_c, \;\; \Omega_c), \tag{27}$$

where

$$G\left(x_C, \;\; \Omega_C\right) = \Omega_C{}^2 x - \alpha\, x_C - \beta\, x_C{}^3 - \lambda\, x_C{}^5 \tag{28}$$

and

$$G_x = \frac{\partial G}{\partial x} = \Omega_C{}^2 - \alpha - 3\beta\, x_C{}^2 - 5\lambda\, x_C{}^4 \tag{29}$$

Here x_C and Ω_C denotes the solutions and the frequencies of a quintic vibrations or cubic-quintic Duffing oscillator.

Applying the proposed method defined in Eq. (6) to Eq. (27), we found

$$
\begin{aligned}
& \ddot{x}_{C_{K+1}} + \Omega_{C_K}^2 x_{C_K} = \left(\Omega_{C_K}^2 x_{C_0} - \alpha x_{C_0} - \beta x_{C_0}^3 - \lambda x_{C_0}^5 \right) \\
& + \left(\Omega_{C_K}^2 - \alpha - 3\beta x_{C_K}^2 - 5\lambda x_{C_K}^4 \right) \left(x_{C_K} - x_{C_0} \right).
\end{aligned}
\tag{30}
$$

For the first iteration, we attain

$$\ddot{x}_{C_1} + \Omega_{C_0}^2 x_{C_1} \;=\; \Omega_{C_0}^2 A \cos\theta \;-\; \alpha\, A \cos\theta - \beta\,(A\cos\theta)^3 - \lambda\,(A\cos\theta)^5 \tag{31}$$

Applying Fourier series in the right side of Eq. (31), it is found

$$
\begin{aligned}
\ddot{x}_{C_1} + \Omega_{C_0}^2 x_{C_1} = & \left(\Omega_{C_1}^2 A \;-\; \alpha\, A - \frac{3}{4}\beta\, A^3 - \frac{5}{8}\lambda\, A^5 \right) \cos\theta \\
& - \left(\frac{1}{4}\beta\, A^3 + \frac{5}{16}\lambda\, A^5 \right) \cos 3\theta - \frac{1}{16}\lambda\, A^5 \cos 5\theta.
\end{aligned}
\tag{32}
$$

By avoiding the secular terms, considering $\cos\theta = 0$. Then we have First approximate frequency,

$$\Omega_{C_0} = \Omega_{M\,0} = \sqrt{\alpha + \frac{3}{4}\beta\, A^2 + \frac{5}{8}\lambda\, A^4} \tag{33}$$

The first approximate frequency of the Eq. (7) is shown in Eq. (33).

After being simplified, the Eq. (32) gives the first approximation of the Eq. (7) is

$$x_{C_1}(t) = f_{C_{11}}\cos\theta + f_{C_{13}}\cos 3\theta + f_{C_{15}}\cos 5\theta, \tag{34}$$

where

$$f_{C_{11}} = \frac{A\left(-3A^2\beta - 4A^4\lambda + 96\Omega^2 \right)}{96\Omega^2}, \tag{35}$$

$$f_{C_{13}} = \frac{A^3\left(4\beta + 5A^2\lambda\right)}{128\Omega^2}, \tag{36}$$

$$f_{C_{15}} = \frac{A^5\lambda}{384\Omega^2}, \Omega^2 = \Omega_{C0}^2. \tag{37}$$

Applying the proposed method for the second iteration, we derive

$$\ddot{x}_{C_2} + \Omega_{C_1}^2 x_{C_2} = f_{C_{21}}\cos\theta + f_{C_{23}}\cos 3\theta + f_{C_{25}}\cos 5\theta, \tag{38}$$

where

$$
\begin{aligned}
f_{C_{21}} = & A\alpha + \frac{3A^3\beta}{4} + \frac{5A^5\lambda}{8} - \frac{A^3\alpha\beta}{32\Omega^2} - \frac{3A^5\beta^2}{64\Omega^2} - \frac{A^5\alpha\lambda}{24\Omega^2} - \frac{29A^7\beta\lambda}{256\Omega^2} - \frac{35A^9\lambda^2}{512\Omega^2} \\
& - A\Omega 1^2 + \frac{A^3\beta\Omega 1^2}{32\Omega^2} + \frac{A^5\lambda\Omega 1^2}{24\Omega^2},
\end{aligned}
\tag{39}
$$

Table 1
Comparison between the Mickens' iteration method and the proposed method of $\ddot{x} + \alpha x + \beta x^3 + \lambda x^5 = 0$, for $\alpha = \beta = \lambda = 1$.

A	Mickens' Iteration Method			Proposed Method			Exact Frequency
	Ω_{M_0} Error (%)	Ω_{M_1} Error (%)	Ω_{M_2} Error (%)	Ω_{C_0} Error (%)	Ω_{C_1} Error (%)	Ω_{C_2} Error (%)	
1	1.541104 1.1498	1.524791 **0.0791193**	1.524466 **0.057757**	1.541104 1.1498	1.522921 **0.0436**	1.52359 **0.00026**	1.523586029

Table 2
Comparison between the Mickens' iteration method and the proposed method of $\ddot{x}_M + \alpha x_M + \beta x_M^3 = 0$, for $\alpha = \beta = 1$.

βA^2	Mickens' Iteration Method			Proposed Method			Exact Frequency
	Ω_{MD_0} Error (%)	Ω_{MD_1} Error (%)	Ω_{MD_2} Error (%)	Ω_{D_0} Error (%)	Ω_{D_1} Error (%)	Ω_{D_2} Error (%)	
1	1.3229 0.3870	1.31799 **0.0144**	1.31795 **0.0114**	1.3229 0.3870	1.3177 **0.0076**	1.3178 **0.00004**	1.3178
100	8.7178 2.1585	8.5539 **0.2379**	8.54812 **0.1702**	8.7178 2.1585	8.5313 **0.0269**	8.5341 **0.0059**	8.5336

Table 3
Comparing of the estimated frequencies with the existing frequencies and the exact frequency of $\ddot{x} + \alpha x + \beta x^3 + \lambda x^5 = 0$, for $\alpha = \beta = \lambda = 1$, and $A = 10$.

$\Omega_{exact}(A = 10) = \Omega_{Ce} = 75.177400$			
Amplitude A	Ω_{C_0} Error (%)	Ω_{C_1} Error (%)	Ω_{C_2} Error (%)
Stiffness Analytical Approaches (SAA) [33]	82.113742 8.447		
Max-Min Technique (MMT), the Iteration Perturbation Method (IPM), and the Homotopy Perturbation Method (HPM) with an auxiliary term [33]	79.536155 5.7979		
He's Frequency-Amplitude Formulation (HFAF) [34]	79.536155 5.7979		
He's Energy Balance Method (HEBM) [34]	76.872188 2.2544		
Frequency Formulation(FF) [16]	72.541942 3.5057		
Modified Homotopy Perturbation Method (MHPM) [6]	79.634795 5.9292		
Energy Balance Method (EBM) [6]	76.999820 2.4242		
Iteration Perturbation Method (IPM) [14]	79.536155 5.7979		
Homotopy Analysis Method (HAM) [31]	79.536155 5.7979		
Newton–Harmonic Balancing Approach (NHBA) [24]	79.5362 5.80	75.9738 1.06	
Coupled Homotopy-Variational Formulation (CHVF) [23]	56.35601 25.04	63.55476 15.46	
Harmonic Balance Method (HBM) with the truncation property [10]	79.536155 5.7979	75.929091 0.9998	
Harmonic Balance Method (HBM) without the truncation property [10]	79.536155 5.7979	76.421355 1.6546	
Proposed method	79.536155 5.7979	**74.825686** **0.4678**	**75.140686** **0.0488**

$$f_{C_{23}} = \frac{A^3\beta}{4} + \frac{5A^5\lambda}{16} + \frac{A^3\alpha\beta}{32\Omega^2} + \frac{3A^5\beta^2}{128\Omega^2} + \frac{5A^5\alpha\lambda}{128\Omega^2} + \frac{5A^7\beta\lambda}{128\Omega^2} + \frac{35A^9\lambda^2}{3072\Omega^2} - \frac{A^3\beta\Omega1^2}{32\Omega^2} - \frac{5A^5\lambda\Omega1^2}{128\Omega^2}, \tag{40}$$

$$f_{C_{25}} = \frac{A^5\lambda}{16} + \frac{3A^5\beta^2}{128\Omega^2} + \frac{A^5\alpha\lambda}{384\Omega^2} + \frac{A^7\beta\lambda}{16\Omega^2} + \frac{125A^9\lambda^2}{3072\Omega^2} - \frac{A^5\lambda\Omega1^2}{384\Omega^2}, \tag{41}$$

By avoiding the secular term, taking $\cos\theta = 0$. Then we have from Eq. (38) and Eq. (39)

$$\Omega_{C_1} = \sqrt{\Omega^2 + \frac{36A^4\beta^2 + 96A^6\beta\lambda + 65A^8\lambda^2}{48A^2\beta + 64A^4\lambda - 1536\Omega^2}}, \tag{42}$$

where $\Omega_{C_1} = \Omega1 =$ second approximate frequency

Eq. (42) represents the second approximate frequency of the Eq. (7).

After being simplified, the second approximate solution of the Eq. (7) is

$$x_{C_2}(t) = f_{C_{31}}\cos\theta + f_{C_{33}}\cos3\theta + f_{C_{35}}\cos5\theta, \tag{43}$$

where

$$\begin{aligned} f_{C_{31}} = A - &\Big(A^{15}\lambda\Big(22590741888\beta^5 + 145021631040A^2\beta^4\lambda \\ &+ 372886443072A^4\beta^3\lambda^2 + 480037894752A^6\beta^2\lambda^3 \\ &+ 309408876114A^8\beta\lambda^4 + 79880562167A^{10}\lambda^5\Big)\Big) \\ &\Big/\Big(1604691041250705408\Omega1^{12}\Big) \\ &+ \frac{1}{241089399226368\Omega1^{10}}A^{13}\lambda\Big(221709312\beta^4 + 1149240312A^2\beta^3\lambda \\ &+ 2237181012A^4\beta^2\lambda^2 + 1938393798A^6\beta\lambda^3 + 630736237A^8\lambda^4\Big) \\ &- \frac{1}{2283043553280\Omega1^8}A^9\Big(16547328\beta^4 + 122580864A^2\beta^3\lambda \\ &+ 313072128A^4\beta^2\lambda^2 + 338191875A^6\beta\lambda^3 + 132612995A^8\lambda^4\Big) \\ &+ \frac{A^7\left(3990\beta^3 + 15694A^2\beta^2\lambda + 20825A^4\beta\lambda^2 + 9315A^6\lambda^3\right)}{13762560\Omega1^6} \\ &- \frac{A^3\left(576\beta\left(\alpha + A^2\beta\right) + 4A^2\left(184\alpha + 285A^2\beta\right)\lambda + 509A^6\lambda^2\right)}{147456\Omega1^4} \\ &- \frac{126A^3\beta + 169A^5\lambda}{4608\Omega1^2}, \end{aligned} \tag{44}$$

Table 4
Comparing of the estimated frequencies with the existing frequencies and the exact frequency of $\ddot{x} + \alpha x + \beta x^3 + \lambda x^5 = 0$, for $\alpha = \beta = \lambda = 1$.

A	$\Omega_{2nd}^{Proposed}$ Error (%)	$\Omega_{3rd}^{Proposed}$ Error (%)	$\Omega_{[10]}^{2nd\ truun}$ Error (%)	$\Omega_{[10]}^{2nd\ no\ truun}$ Error (%)	$\Omega_{[14]}^{HEBM}$ Error (%)	$\Omega_{[15]}^{IPM}$ Error (%)	$\Omega_{[24]}^{ex}$
0.10	**1.003773** **0.0002**	1.003773 **0.0002**	1.003772 0.0001	1.003772 0.0001	1.003773 0.0002	1.003774 0.0003	1.003770
0.30	**1.035539** **0.0000002**	1.035539 **0.0000002**	1.035540 0.00009	1.035540 0.00009	1.035492 0.0046	1.035645 0.0101	1.035539
0.50	**1.106535** **0.0004**	1.106544 **0.0003**	1.106578 0.0034	1.106591 0.0046	1.106356 0.0166	1.107502 0.0869	1.106540
1.0	**1.522921** **0.0436**	1.523590 **0.00026**	1.525150 0.1023	1.526041 0.1608	1.527720 0.2710	1.541103 1.1494	1.523586
3.0	**7.241105** **0.3781**	7.266293 **0.03215**	7.327622 0.8115	7.367509 1.3603	7.417768 2.0518	7.640353 5.1140	7.268630
5.0	**19.096925** **0.4409**	19.171557 **0.05184**	19.362576 0.9440	19.482606 1.5697	19.60879 2.2276	20.257714 5.6106	19.18150
8.0	**48.054014** **0.4982**	48.271453 **0.04793**	48.772419 0.9893	49.086138 1.6389	49.39062 2.2694	51.078371 5.7641	48.29460
10.0	**74.825686** **0.4678**	**75.140686** **0.0488**	75.929091 0.9998	76.421355 1.6546	76.88958 2.2775	79.536155 5.7979	75.17740

Note: Here $\Omega_{2nd}^{Proposed}$, $\Omega_{3rd}^{Proposed}$ represent the second and third approximate frequencies of (7) obtained by the adopted method, $\Omega_{[10]}^{2nd\ truun}$ (using the truncation formula) and $\Omega_{[10]}^{2nd\ no\ truun}$ (without the truncation formula) represent the second approximation frequencies obtained by Chowdhury *et al.* [10], respectively. $\Omega_{[14]}^{HEBM}$ and $\Omega_{[15]}^{IPM}$ represent the second approximate frequencies obtained by Ganji *et al.* [14,15]. Ω_{ex} is the exact frequency obtained by Lai et al. [24] and $Error(\%) = \left|\frac{\Omega_{ex} - \Omega_k}{\Omega_{ex}}\right| \times 100$, $k = 0, 1, \cdots\cdots$ is the percentage Error.

Table 5
Comparing of the estimated frequencies with existing frequencies and the exact frequency of $\ddot{x} + \alpha x + \beta x^3 = 0$, for $\alpha = \beta = 1$.

βA^2	$\Omega_{2nd}^{Proposed}$ Error (%)	$\Omega_{3rd}^{Proposed}$ Error (%)	$\Omega_{[41]}^{2nd\ MEBM}$ Error (%)	$\Omega_{[3]}^{2nd\ MEBM}$ Error (%)	$\Omega_{[13]}^{2nd\ HEBM}$ Error (%)	$\Omega_{[12]}^{2nd\ EBM}$ Error (%)	Ω_{ex}
0.5	**1.1708** **0.00144**	**1.1708** **0.00004**	1.1708 **0.00144**	1.1708 **0.00144**	1.1702 0.0512	1.1702 0.0512	1.1708
1	**1.3177** **0.004898**	**1.3178** **0.00026**	1.3180 0.0152	1.3180 0.0151	1.3161 0.1290	1.3163 0.1138	1.3178
5	**2.1500** **0.0186**	**2.1505** **0.0047**	2.1528 0.1116	2.1518 0.0651	2.1406 0.4557	2.1426 0.3627	2.1504
10	**2.8660** **0.0209**	**2.8667** **0.0035**	2.87133 0.1631	2.8690 0.0837	2.8500 05790	2.8535 0.4569	2.8666
100	**8.5314** **0.0258**	**8.5341** **0.0059**	8.5539 0.2379	8.5425 0.1042	8.4700 0.7452	8.4261 1.2597	8.5336
1000	**26.8037** **0.0261**	**26.8124** **0.0063**	26.8773 0.2448	26.8394 0.1070	26.6055 0.7653	26.6519 0.5923	26.8107
5000	**59.8999** **0.0264**	**59.9196** **0.0065**	60.0650 0.2499	59.9799 0.1071	59.4559 0.7674	59.5599 0.5938	59.9157

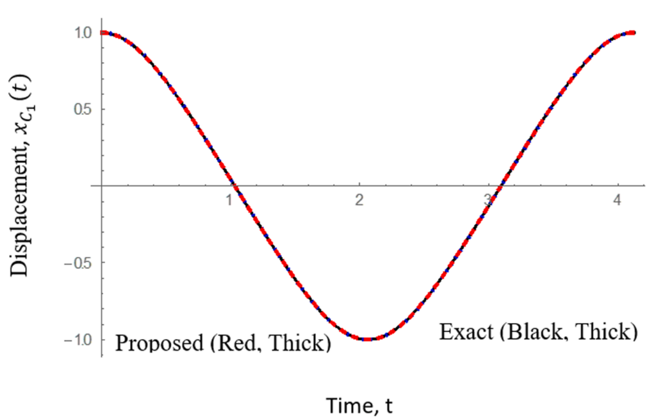

Fig. 2. Comparison of the numerical solutions and the second-order approximations of $\ddot{x} + \alpha x + \beta x^3 + \lambda x^5 = 0$, for $\alpha = \beta = \lambda = 1$, and A=1.

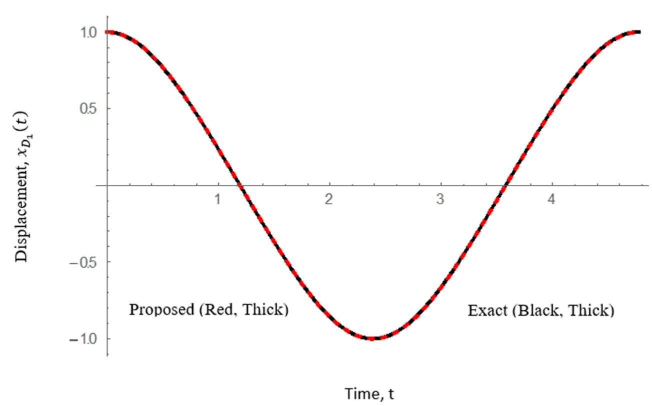

Fig. 3. Comparison of the numerical solutions and the second-order approximations of $\ddot{x} + \alpha x + \beta x^3 = 0$, for α=β=1, and A=1.

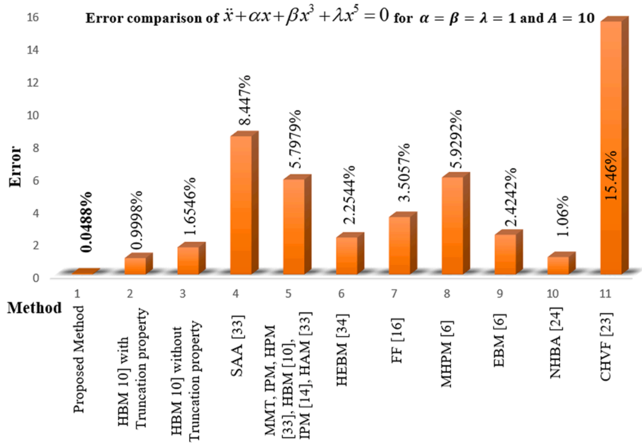

Fig. 4. Error comparison of $\ddot{x}+\alpha x+\beta x^3+\lambda x^5=0$, for $\alpha=\beta=\lambda=1$, and A=10, among the suggested method and the several existing methods.

$$f_{C_{33}}=\frac{1}{267181325549568\Omega1^{12}}A^3\Big(5A^{12}\lambda\big(808704\beta^5+5152896A^2\beta^4\lambda+13148352A^4\beta^3\lambda^2+16794288A^6\beta^2\lambda^3+10738014A^8\beta\lambda^4+2749475A^{10}\lambda^5\big)-960A^{10}\lambda\big(259200\beta^4+1330560A^2\beta^3\lambda+2564352A^4\beta^2\lambda^2+2199120A^6\beta\lambda^3+708043A^8\lambda^4\big)\Omega1^2+147456A^6\big(13824\beta^4+95904A^2\beta^3\lambda+234468A^4\beta^2\lambda^2+244937A^6\beta\lambda^3+93385A^8\lambda^4\big)\Omega1^4-28311552A^4\big(2592\beta^3+9504A^2\beta^2\lambda+11676A^4\beta\lambda^2+4805A^6\lambda^3\big)\Omega1^6+10871635968\big(24\beta\big(4\alpha+3A^2\beta\big)+120A^2\big(\alpha+A^2\beta\big)\lambda+35A^6\lambda^2\big)\Omega1^8+1826434842624\big(4\beta+5A^2\lambda\big)\Omega1^{10}\Big), \tag{45}$$

$$f_{C_{35}}=\frac{1}{267181325549568\Omega1^{12}}A^5\Big(A^{10}\lambda\big(-20736\beta^5+120960A^2\beta^4\lambda+967680A^4\beta^3\lambda^2+2094960A^6\beta^2\lambda^3+1899830A^8\beta\lambda^4+632827A^{10}\lambda^5\big)-320A^8\lambda\big(51840\beta^4+314496A^2\beta^3\lambda+701568A^4\beta^2\lambda^2+685440A^6\beta\lambda^3+248311A^8\lambda^4\big)\Omega1^2+49152A^6\lambda\big(23760\beta^3+97380A^2\beta^2\lambda+133119A^4\beta\lambda^2+60695A^6\lambda^3\big)\Omega1^4-9437184A^2\big(864\beta^3+5472A^2\beta^2\lambda+9954A^4\beta\lambda^2+5605A^6\lambda^3\big)\Omega1^6+3623878656\big(72\beta^2+8\big(\alpha+24A^2\beta\big)\lambda+125A^4\lambda^2\big)\Omega1^8+666793672704\lambda\Omega1^{10}\Big). \tag{46}$$

Applying the proposed method and then after being simplified, we obtain the third approximate frequency,

$$\Omega_{C_2}=\frac{1}{1179648\sqrt{3}}\Bigg(\sqrt{\Bigg(\frac{1}{\mathrm{f1}\Omega1^{10}}\Big(5A^{15}\lambda\big(279936\beta^5+1814400A^2\beta^4\lambda+4711392A^4\beta^3\lambda^2+6126624A^6\beta^2\lambda^3+3989811A^8\beta\lambda^4+1040963A^{10}\lambda^5\big)-120A^{12}\lambda\big(51840\big(16A-4\mathrm{f1}+5\mathrm{g1}+\mathrm{h1}\big)\beta^4+24192A^2\big(180A-45\mathrm{f1}+55\mathrm{g1}+13\mathrm{h1}\big)\beta^3\lambda+6048A^4\big(1420A-355\mathrm{f1}+424\mathrm{g1}+116\mathrm{h1}\big)\beta^2\lambda^2+336A^6\big(22444A-5611\mathrm{f1}+85\big(77\mathrm{g1}+24\mathrm{h1}\big)\big)\beta\lambda^3+7A^8\big(355368A-88842\mathrm{f1}+101149\mathrm{g1}+35473\mathrm{h1}\big)\lambda^4\big)\Omega1^2+36864A^9\big(10368\beta^4+864A\big(147A+25\big(-2\mathrm{f1}+2\mathrm{g1}+\mathrm{h1}\big)\big)\beta^3\lambda+216A^3\big(1846A-800\mathrm{f1}+775\mathrm{g1}+425\mathrm{h1}\big)\beta^2\lambda^2+9A^5\big(53881A-25650\mathrm{f1}+24050\mathrm{g1}+14390\mathrm{h1}\big)\beta\lambda^3+5A^7\big(41200A-20600\mathrm{f1}+18677\mathrm{g1}+12139\mathrm{h1}\big)\lambda^4\big)\Omega1^4-3538944A^6\big(864\big(6A-3\mathrm{f1}+3\mathrm{g1}+\mathrm{h1}\big)\beta^3+288A^2\big(108A-69\mathrm{f1}+53\mathrm{g1}+39\mathrm{h1}\big)\beta^2\lambda+6A^4\big(9380A-6650\mathrm{f1}+4426\mathrm{g1}+4339\mathrm{h1}\big)\beta\lambda^2+5A^6\big(6416A-4812\mathrm{f1}+2883\mathrm{g1}+3363\mathrm{h1}\big)\lambda^3\big)\Omega1^6+5435817984A^4\big(36\big(2A-2\mathrm{f1}+\mathrm{g1}+\mathrm{h1}\big)\beta^2+3A^2\big(83A-83\mathrm{f1}+25\mathrm{g1}+47\mathrm{h1}\big)\beta\lambda+5A^4\big(7\big(6A-6\mathrm{f1}+\mathrm{g1}\big)+25\mathrm{h1}\big)\lambda^2\big)\Omega1^8+260919263232\big(16\mathrm{f1}\alpha+12A^2\big(-2A+3\mathrm{f1}+\mathrm{g1}\big)\beta+5A^4\big(-8A+10\mathrm{f1}+5\mathrm{g1}+\mathrm{h1}\big)\lambda\big)\Omega1^{10}\Big)\Bigg)}\Bigg) \tag{47}$$

where $\Omega 1 = \Omega_{C_1}$, $f1 = f_{C_{31}}$, $g1 = f_{C_{33}}$, $h1 = f_{C_{35}}$.

3B.Solution Procedure: Duffing oscillator

Duffing oscillator is a special case of the Eq. (7). If λ=0, then Eq. (7) represents

$\ddot{x} + \alpha\, x + \beta\, x^3 = 0$, with the initial condition

$$x(0) = A, \dot{x}(0) = 0. \tag{48}$$

Equation (48) is called the Duffing oscillator [41]. Equation (48) is also a special case of Duffing [38] equation. Equation (48) is also marked as the free undamped vibration of an orthotropic claimed triangular plate by Askari et al. [4].

Mickens Iteration Method

Since λ=0, the first, second, and third approximate frequencies of the oscillator (48) have been calculated using Eqs. (14), (21) and (26).

$$\Omega_{MD_0} = \text{First approximate frequency} = \sqrt{\alpha + \frac{3}{4}\beta A^2} \tag{49}$$

$$\Omega_{MD_1} = \frac{1}{2}\sqrt{\frac{3}{2}}\sqrt{\frac{\left(4\alpha + 3A^2\beta\right)^2\left(32\alpha + 23A^2\beta\right)\left(2048\alpha^3 + 4608A^2\alpha^2\beta + 3408A^4\alpha\beta^2 + 831A^6\beta^3\right)}{\left(8\alpha + 6A^2\beta\right)^4\left(96\alpha + 69A^2\beta\right)}}, \tag{50}$$

where Ω_{M_1} = second approximate frequency.

$$\Omega_{MD_2} = \frac{1}{294912\sqrt{2}}\sqrt{\frac{-764411904A^8\beta^4\Omega 1^4 + 36691771392A^6\beta^3\Omega 1^6 - 10871635968A^2\left(48\alpha\beta + 72A^2\beta^2\right)\Omega 1^8 + 2087354105856\left(8\alpha + 6A^2\beta\right)\Omega 1^{10}}{\Omega 1^8\left(-3A^2\beta + 96\Omega 1^2\right)}} \tag{51}$$

where $\Omega 1 = \Omega_{MD_1}$, Ω_{MD_2} = Third approximate frequency.

And, the Eqs. (15) and (22) give the first and second approximation of the oscillator (48).

$$x_{MD_1}(t) = f_{MD}\cos\theta + g_{MD}\cos 3\theta, \tag{52}$$

where

$$f_{MD} = \frac{A\left(-3A^2\beta + 96\Omega^2\right)}{96\Omega^2}, g_{MD} = \frac{4A^3\beta}{128\Omega^2}, \Omega^2 = \Omega^2_{M_0}. \tag{53}$$

$$x_{MD_2}(t) = f_{MD_{31}}\cos\theta + f_{MD_{33}}\cos 3\theta + f_{MD_{35}}\cos 5\theta, \tag{54}$$

$$f_{MD_{31}} = -\frac{1}{80234552062535270400\Omega 1^{12}}\Big(A\Big(58153400008704A^8\beta^4\Omega 1^4 - 2326136000348160A^6\beta^3\Omega 1^6 + 31341621899427840A^4\beta^2\Omega 1^8 - 80234552062535270400\Omega 1^{12} + 31341621899427840A^2\beta\Omega 1^8\left(\alpha + 8\Omega 1^2 - \Omega 2^2\right)\Big)\Big), \tag{55}$$

$$f_{MD_{32}} = \frac{1}{267181325549568\Omega 1^{12}}\Big(A^3\Big(2038431744A^6\beta^4\Omega 1^4 - 73383542784A^4\beta^3\Omega 1^6 + 782757789696A^2\beta^2\Omega 1^8 + 1043677052928\beta\Omega 1^8\left(\alpha + 8\Omega 1^2 - \Omega 2^2\right)\Big)\Big), \tag{56}$$

$$f_{MD_{33}} = \frac{A^5\left(-8153726976A^2\beta^3\Omega 1^6 + 260919263232\beta^2\Omega 1^8\right)}{267181325549568\Omega 1^{12}}. \tag{57}$$

Proposed Method

Since λ=0, the first, second, and third approximate frequencies of the oscillator (48) have been calculated using Eqs. (33), (42) and (47).

$$\Omega_{D_0} = \sqrt{\alpha + \frac{3}{4}\beta A^2}. \tag{58}$$

$$\Omega_{D_1} = \sqrt{\Omega^2 + \frac{36A^4\beta^2}{48A^2\beta - 1536\Omega^2}}, \tag{59}$$

where $\Omega_{D_1} = \Omega 1$ = second approximate frequency.

$$\begin{aligned}\Omega_{D_2} = \frac{1}{32}\sqrt{}\Big(\Big(&115292150460684697600\alpha^{13} + 1098878309078401024000A^2\alpha^{12}\beta + 4814798361621797273600A^4\alpha^{11}\beta^2 \\ &+ 12840415207735531929600A^6\alpha^{10}\beta^3 + 23253358814266277232640A^8\alpha^9\beta^4 + 30196054006584165007360A^{10}\alpha^8\beta^5 + 28926659002164660142080A^{12}\alpha^7\beta^6 \\ &+ 20697637207599900262400A^{14}\alpha^6\beta^7 + 11061585792524972195840A^{16}\alpha^5\beta^8 + 4361107334070502359040A^{18}\alpha^4\beta^9 + 1232913957664921886720A^{20}\alpha^3\beta^{10} \\ &+ 236686156489278643200A^{22}\alpha^2\beta^{11} + 27655639266968495280A^{24}\alpha\beta^{12} + 1485496121097756690A^{26}\beta^{13}\Big)\Big/\Big(\left(64\alpha^2 + 94A^2\alpha\beta + 33A^4\beta^2\right)^2 \\ &\Big(2748779069440\alpha^8 + 16063177687040A^2\alpha^7\beta + 40804873666560A^4\alpha^6\beta^2 + 58845791518720A^6\alpha^5\beta^3 + 52690059001856A^8\alpha^4\beta^4 \\ &+ 29994580361216A^{10}\alpha^3\beta^5 + 10601612691456A^{12}\alpha^2\beta^6 + 2127328259256A^{14}\alpha\beta^7 + 185567889521A^{16}\beta^8\Big)\Big)\Big).\end{aligned} \tag{60}$$

where $\Omega_{D_2} = \Omega 2 =$ Third approximate frequency.

And, the Eqs. (34) and (43) give the first and second approximate solutions of the oscillator (48).

$$x_{D_1}(t) = f_{D_{11}}\cos\theta + f_{D_{13}}\cos 3\theta, \tag{61}$$

where

$$f_{D_{11}} = \frac{A(-3A^2\beta + 96\Omega^2)}{96\Omega^2}, f_{D_{13}} = \frac{4A^3\beta}{128\Omega^2}, \Omega^2_{M_0} = \Omega^2. \tag{62}$$

$$x_{D_2}(t) = f_{D_{31}}\cos\theta + f_{D_{33}}\cos 3\theta + f_{D_{35}}\cos 5\theta. \tag{63}$$

where

$$f_{D_{31}} = A - \frac{1}{2283043553280\Omega 1^8}A^9\left(16547328\beta^4\right) + \frac{A^7(3990\beta^3)}{13762560\Omega 1^6} - \frac{A^3(576\beta(\alpha + A^2\beta))}{147456\Omega 1^4}, \tag{64}$$

$$f_{D_{33}} = \frac{1}{267181325549568\Omega 1^{12}}A^3\left(147456A^6(13824\beta^4)\Omega 1^4 - 28311552A^4(2592\beta^3)\Omega 1^6 + 10871635968(24\beta(4\alpha + 3A^2\beta))\Omega 1^{10}\right), \tag{65}$$

$$f_{D_{35}} = \frac{1}{267181325549568\Omega 1^{12}}A^5\left(-9437184A^2\left(864\beta^3\right)\Omega 1^6 + 3623878656\left(72\beta^2\right)\Omega 1^8\right) \tag{66}$$

4. A discussion of the findings

The Duffing equation and an equation of motion of nonlinear vibrations are both successfully solved using the Modified Mickens Extended Iteration method. The computed solutions developed by the proposed approach are in good agreement with the results obtained through other existing analytical methods including Mickens' iteration method. Obtained results are more trustworthy, especially when dealing with higher-order nonlinearities. It has been compared to the Mickens Iteration Method in Tables 1 and 2 to prove the independence of the proposed method. Table 3 compares the approximate frequencies of (7) with the results obtained by the Stiffness analytical approaches [33], the homotopy perturbation method with an auxiliary term [33], the max-min approach [33], the iteration perturbation method [33], the He's frequency-amplitude formulation [34], the He's energy balance method [34], the one-step frequency formulation [16], the modified homotopy perturbation method [6], the energy balance method [6], the iteration perturbation method [14,15], the homotopy analysis method [31], the Newton-harmonic balancing approach, the coupled homotopy-variational formulation [23], the harmonic balance method with truncation property [10], and without truncation property [10]. The frequency-amplitude relationship of (7) has been shown in Table 4, and it has been compared with other existing methods, including the harmonic balance method with the truncation property [10], the harmonic balance method without the truncation property [10], the iteration perturbation method [14], and He's energy Balance Method [15]. The frequency-amplitude relationship of Duffing oscillator has been shown in Table 5, and it has been compared with other approaches including the modified energy balance method [3], the energy balance method [12], a modified Mickens iteration procedure [41], and the He's energy balance method [13]. The exact solutions and the second-order approximate solutions of (7) for α=β=λ=1 and A=1 are compared in Fig. 2. The exact solutions and the second-order approximate solutions of the Duffing equation (48) for α=β=1 and A=1 are compared in Fig. 3. Fig. 4 compares the error made by the suggested method to the errors obtained using several existing methods.

Here, $\Omega^{Proposed}_{2nd}$, $\Omega^{Proposed}_{3rd}$ represent the second and third approximate frequencies of (48) obtained by the proposed method, $\Omega^{2nd\ MEBM}_{[41]}$, $\Omega^{2nd\ MEBM}_{[3]}$, $\Omega^{2nd\ HEBM}_{[13]}$ *and* $\Omega^{2nd\ EBM}_{[12]}$ and represent the second approximation frequencies obtained by Chen & Liu [41], Hosen *et al.* [3], Durmaz *et al.* [12,13] respectively. Ω_{ex} is the exact frequency obtained by Durmaz *et al.* [13] and Er(%) is the percentage error defined as above.

5. Conclusion

The results of the approximate frequencies, especially the third estimated frequencies, give very good results that are extremely close to the exact results, and the obtained results are better than the existing results. Additionally, when compared to other existing methods, the solution processes of the suggested method for both oscillators are simple. Finally, we summarized:

- The suggested method, the modified Mickens extended iteration method, provides a
- potent method for studying random oscillations. Also, it is a quick and effective way to determine the approximate frequencies and the periodic solutions of nonlinear oscillators.
- The suggested method yields ideal values for each small and large value of the initial amplitude. While most approaches produce less inaccuracy for smaller values of the initial amplitude and more for larger ones.
- The suggested method yields outstanding results for higher-order solutions, whereas the majority of methods produce decent results for first-order solutions but poor results for higher-order solutions.
- The suggested method is really uncomplicated, direct, and transparent.
- The suggested method is better than other existing methods including Mickens iteration method.
- An equation of motion of nonlinear vibrations has a 0.0488% error for the third approximate frequency when the initial amplitude is set to 10 and the values of the real system constant parameters are set to 1. The Duffing oscillator has a 0.00026% error for the third approximate frequency when the initial amplitude is 1.
- We have also concluded that it is not mandatory to use higher-order harmonic terms in every iteration step to obtain a better improvement.

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